

The University of Chicago

CERTAIN TYPES OF NILPOTENT
ALGEBRAS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By

FANNIE WILSON BOYCE

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CERTAIN TYPES OF NILPOTENT ALGEBRAS

INTRODUCTION

The purpose of this paper is the classification of linear associative nilpotent algebras of orders less than 6 and the generalization to order n of certain types of algebras which arise. In particular, we consider algebras of order n of degrees n , $n - 1$, $n - 2$, and $n - 3$.

The terms field, algebra, index, nilpotent, order, basis, equivalence of algebras, are defined as in Dickson's Algebren und ihre Zahlentheorie.

The degree of an element of a nilpotent algebra is defined to be the exponent of the highest power of the element which is different from zero. The degree of the algebra is that of an element of highest degree in it.¹ It is evident that nilpotent algebras of different degrees are inequivalent.

If N is a nilpotent algebra of index a , evidently the linear sets $B_i \equiv N^i - N^{i+1}$ ($i = 1, \dots, a - 1$) have an invarientive definition. We shall use the invariance of these sets in the classification of the algebras.

The letters e_i ($i = 1, \dots, n$) will be used throughout the paper to denote the basal units of an algebra of order n . All other letters, used as coefficients of the e_i , are understood to represent quantities of the coefficient field F . If every

¹H. E. Hawkes, On hypercomplex number systems, Transactions of the American Mathematical Society, vol. 3 (1902), p. 321.

product of two basal units is of the form $e_i e_j = \sum_{k=1}^n a_{ijk} e_k$ where $s > i, s > j$ and the a_{ijk} are in F , then the basal units are said to be normalized. Such a basis always exists.¹

The multiplication tables given in the classification in Chapter VII do not in general define algebras, but classes of algebras. For different values of the coefficients we may have inequivalent members of the same class. Throughout this paper we shall refer to these classes as algebras. For example, we shall speak of algebra 13 in n units, although algebra 13 is in reality a class of algebras. Zero algebras and reducible algebras are omitted, as well as the reciprocal of an algebra, even if it is inequivalent to the algebra itself. If a product $e_i e_j$ is zero, it will generally be omitted from the multiplication table, but we shall sometimes note the fact that a particular product vanishes in the equations leading up to our final forms.

¹L. E. Dickson, Algebren und ihre Zahlentheorie, 1927, p. 111.

CHAPTER I

FUNDAMENTAL LEMMAS OF THE THEORY

It has been proved that the basal units of a nilpotent algebra can always be so chosen that they are normalized. We shall show how this choice affects the multiplication tables of algebras of order 3. For this purpose, we first prove the trivial

THEOREM 1. In a nilpotent algebra two of whose basal units are e_i and e_j , $e_i e_j = a e_i$ implies that $a = 0$. Similarly $e_i e_j = b e_j$ implies that $b = 0$.

Let e_j be of degree $r - 1$. Now suppose that $e_i e_j = a e_i$, $a \neq 0$. Then $e_i e_j^2 = a e_i e_j = a^2 e_i$, $e_i e_j^3 = a^2 e_i e_j = a^3 e_i$, ..., $e_i e_j^{r-1} = a^{r-1} e_i$, and finally $0 = e_i e_j^r = a^r e_i \neq 0$. This contradiction shows that $a = 0$. Similarly $e_i e_j = b e_j$ implies $b = 0$.

We next prove

THEOREM 2. In a nilpotent algebra of order n and degree $n - 1$, let the basal units be so chosen that $e_2^2 = e_3$, $e_2^3 = e_4$, ..., $e_2^{n-1} = e_n$, while e_1 is independent of the powers of e_2 . Then $e_1 e_n = e_n e_1 = 0$ ($i = 1, \dots, n$).

First let $i \geq 2$. Then $e_1 e_n = e_2^{i-1} e_2^{n-1} = e_2^{n+i-2} = e_2^r$ where $r \geq n$. Hence $e_1 e_n = 0$ ($i = 2, \dots, n$). Similarly, $e_n e_1 = 0$ ($i = 2, \dots, n$). Now

$$(1) \quad e_1 e_n = c_1 e_1 + c_2 e_2 + \dots + c_n e_n.$$

Hence $0 = e_1 e_n^2 = c_1 e_1 e_n + c_2 e_2 e_n + \dots + c_n e_n^2 = c_1 e_1 e_n$, the other products being zero by the first part of the proof. Therefore either $c_1 = 0$ or $e_1 e_n = 0$. But it is evident from (1) that

if $e_1 e_n = 0$, then $c_1 = 0$ also. Hence in any case $c_1 = 0$ and $e_1 e_n = c_2 e_2 + c_3 e_3 + \dots + c_n e_n$. Then $0 = e_1 e_n e_2 = c_2 e_2^2 + c_3 e_3 e_2 + \dots + c_{n-1} e_{n-1} e_2 + e_n e_n e_2 = c_2 e_3 + c_3 e_4 + \dots + c_{n-1} e_n$. Therefore $c_2 = c_3 = \dots = c_{n-1} = 0$ and $e_1 e_n = c_n e_n$. But by Theorem 1, $c_n = 0$, and hence $e_1 e_n = 0$. Similarly $e_n e_1 = 0$.

We shall also need

THEOREM 3. In a nilpotent algebra of order $n > 2$ and degree $n - 1$ with basal units chosen as in Theorem 2, $e_1 e_{n-1} = d_n e_n$ and $e_{n-1} e_1 = g_n e_n$.

From $e_1 e_{n-1} = d_1 e_1 + d_2 e_2 + \dots + d_n e_n$, we have $0 = e_1 e_{n-1} e_2^{n-2} = d_1 e_1 e_2^{n-2} + d_2 e_2 e_2^{n-2} + \dots + d_n e_n e_2^{n-2} = d_1 e_1 e_{n-1} + d_2 e_n$. Therefore $d_1 e_1 e_{n-1} = -d_2 e_n = d_1 (d_1 e_1 + d_2 e_2 + \dots + d_n e_n)$. Hence $d_1^2 = 0$, $d_1 = 0 = d_2$ and $e_1 e_{n-1} = d_3 e_3 + \dots + d_n e_n$. Then $e_1 e_{n-1} e_2 = e_1 e_n = 0 = d_3 e_3 e_2 + d_4 e_4 e_2 + \dots + d_n e_n e_2 = d_3 e_4 + d_4 e_5 + \dots + d_{n-1} e_n$. Consequently, $d_3 = d_4 = \dots = d_{n-1} = 0$ and $e_1 e_{n-1} = d_n e_n$. Similarly, $e_{n-1} e_1 = g_n e_n$.

We can now prove

THEOREM 4. In a nilpotent algebra of order 3 and degree 2, if the basal units are so chosen that $e_2^2 = e_3$, e_1 being independent of e_2 and e_2^2 , the basal units are normalized.

Since $e_1^2 = a_{11} e_1 + b_{11} e_2 + c_{11} e_3$, we have $0 = e_1^3 = a_{11} e_1^2 + b_{11} e_1 e_2 + c_{11} e_1 e_3 = a_{11} e_1^2 + b_{11} c_{12} e_3$ by Theorems 2 and 3. Hence $a_{11} e_1^2 = -b_{11} c_{12} e_3 = a_{11} (a_{11} e_1 + b_{11} e_2 + c_{11} e_3)$, $0 = a_{11}^2 e_1 + a_{11} b_{11} e_2 + (a_{11} c_{11} + b_{11} c_{12}) e_3$ and $a_{11}^2 = 0$. Then $e_1^2 = b_{11} e_2 + c_{11} e_3$. By Theorem 3, $e_1 e_2 = c_{12} e_3$, $e_2 e_1 = c_{21} e_3$. By Theorem 2, $e_1 e_3 = e_3 e_1 = 0$ ($i = 1, 2, 3$). Hence the basal units are normalized.

After the multiplication tables of all nilpotent algebras of order 3 have been so determined that the basal units are normalized, we shall be able to find the multiplication tables of

all nilpotent algebras of order 4, then of order 5, and so on, by means of the following theorem. Note that the basal units in all the derived algebras will necessarily be normalized.

THEOREM¹ 5. The multiplication tables of a set of nilpotent algebras in n units $e_1, \dots, e_n$, over a field F , to which every linear associative nilpotent algebra of order n over F is either equivalent or reciprocal, can be found from a set of the same kind in $n - 1$ units and multiplication tables of the form

$$e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k \quad (i, j = 1, \dots, n-1)$$

by writing

$$e_i e_n = e_n e_i = 0 \quad (i = 1, \dots, n),$$

$$e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k + c_{ijn} e_n \quad (i, j = 1, \dots, n-1),$$

where the $(n-1)^2$ quantities c_{ij} in F are to be so determined that the algebras obtained are associative.

It will usually be convenient to use instead of Theorem 5 one of three consequences we shall derive from it. We shall frequently use the case $s = n - 1$ of

THEOREM 6. Let N be a linear associative nilpotent algebra of order $n - 1$ and of degree $s - r + 1$, having the multiplication table

$$e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k \quad (i < r \text{ or } i > s, j = 1, \dots, n-1, \text{ and } i = 1, \dots, n-1, j < r \text{ or } j > s),$$

$$(2) \quad e_r^\ell = e_{r+\ell-1} \quad (\ell = 1, 2, \dots, s - r + 1, r < s \leq n - 1).$$

¹R. B. Allen, Number systems in an arbitrary domain, Transactions of the American Mathematical Society, vol. 9 (1908), p. 213.

Then every linear associative nilpotent algebra of order n derived from N by the use of Theorem 5 has either the multiplication table

$$(3) \quad e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k + c_{ij} e_n \quad (1 < r \text{ or } 1 > s, j = 1, \dots, n-1, \\ \text{and } 1 = 1, \dots, n-1, j < r \text{ or } j > s),$$

$$(4) \quad e_r^{s-r+2} = e_n,$$

$$(5) \quad e_1 e_n = e_n e_1 = 0 \quad (1 = 1, \dots, n),$$

together with (2), or the multiplication table (3), (2), (5) and

$$(6) \quad e_r^{s-r+2} = 0.$$

For by Theorem 5 the multiplication tables of the new algebras derived from N may be written in the form (3), (5) and

$$(7) \quad e_i e_j = e_{i+j-r+1} + c_{ij} e_n \quad (r \leq i \leq s, r \leq j \leq s)$$

where $e_{i+j-r+1} = 0$ for $i + j - r + 1 > s$. Now put

$$(8) \quad e_j' = e_j + c_{r,j-1} e_n \quad (j = r + 1, \dots, s).$$

Then $e_{r+1}' = e_{r+1} + c_{rr} e_n$ and from (7) we have $e_r^2 = e_{r+1} + c_{rr} e_n = e_{r+1}'$. Then $e_r^3 = e_r e_{r+1}' = e_r e_{r+1} = e_{r+2} + c_{r,r+1} e_n = e_{r+2}' = e_{r+1}' e_r = e_{r+1} e_r = e_{r+2} + c_{r+1,r} e_n$. Hence $c_{r,r+1} = c_{r+1,r}$. Continuing this process we have $e_r^{s-r+1} = e_r e_r^{s-r} = e_r e_{s-1}' = e_r e_{s-1} = e_s + c_{r,s-1} e_n = e_s' = e_r^{s-r} e_r = e_{s-1}' e_r = e_{s-1} e_r = e_s = c_{s-1,r} e_n$, and $c_{r,s-1} = c_{s-1,r}$. Finally, $e_r^{s-r+2} = e_r e_s' = e_r e_s = c_{rs} e_n = e_s' e_r = e_s e_r = c_{sr} e_n$ and $c_{rs} = c_{sr}$. We have shown that since the algebras are associative we must have

$$(9) \quad c_{rj} = c_{jr} \quad (j = r + 1, \dots, s).$$

If $c_{rs} \neq 0$, put $e_n' = c_{rs}e_n$. Then the basal units defined by (8) with $e_r' = e_r$ and $e_n' = c_{rs}e_n$ satisfy (2) and (4). In case $c_{rs} = 0$, the basal units (8) and $e_r' = e_r$ satisfy (2) and (6).

It is evident from (8) and (5) that

$$e_i e_j' = e_i e_j, \quad e_j' e_i = e_j e_i \quad (i < r \text{ or } i > s, j = r+1, \dots, s).$$

Hence the transformation (8) leaves (3) unaltered. When $c_{rs} \neq 0$, the transformation $e_n' = c_{rs}e_n$ merely replaces the c_{ij} in (3) by $c_{ij}' = c_{rs}^{-1}c_{ij}$. We have now shown that if $c_{rs} \neq 0$, the new algebras have the multiplication tables (3), (2), (4), (5), while for $c_{rs} = 0$, they have the multiplication tables (3), (2), (6), (5).

Using (8), (5), (7) and (9) we find $c_{r+p, r+q} = c_{r, r+p+q} = c_{r+p+q, r}$ for $p \geq 1, q \geq 1, r + p + q \leq s$.

We shall also need a result similar to Theorem 6, which we state as

THEOREM 7. Let N be a linear associative nilpotent algebra of order $n - 1$ and of degree $s - r$, having the multiplication table

$$\begin{aligned} e_i e_j &= \sum_{k=1}^{n-1} c_{ijk} e_k \quad (i < r \text{ or } i > s \text{ or } i = r+1, j = 1, \dots, n-1, \\ &\quad \text{and } i = 1, \dots, n-1, j < r \text{ or } j > s \text{ or } j = r+1), \\ (10) \quad e_r^\ell &= e_{r+\ell} \quad (\ell = 2, 3, \dots, s - r; r < s \leq n - 1). \end{aligned}$$

Then every algebra of order n derived from N by the use of Theorem 5 has either the multiplication table

$$\begin{aligned} e_i e_j &= \sum_{k=1}^{n-1} c_{ijk} e_k + c_{ij} e_n \\ (11) \quad & \quad (i < r \text{ or } i > s \text{ or } i = r+1, j = 1, \dots, n-1, \text{ and} \\ & \quad i = 1, \dots, n-1, j < r \text{ or } j > s \text{ or } j = r+1), \end{aligned}$$

$$(12) \quad e_r^{s-r+1} = e_n$$

with (10) and (5), or the multiplication table (11), (10), (5), together with

$$(13) \quad e_r^{s-r+1} = 0.$$

For Theorem 5 states that the multiplication tables of the new algebras may be written in the form (11), (5),

$$(14) \quad e_r^2 = e_{r+2} + c_{rr}e_n,$$

$$(15) \quad \begin{aligned} e_1e_j &= e_{1+j-r} + c_{1j}e_n \quad (r+2 \leq 1 \leq s, \quad r+2 \leq j \leq s) \text{ for } 1+j \leq r+s, \\ &= c_{1j}e_n \quad \text{for } 1+j > r+s, \end{aligned}$$

$$(16) \quad \begin{aligned} e_re_j &= e_{j+1} + c_{rj}e_n, \\ e_je_r &= e_{j+1} + c_{jr}e_n \end{aligned} \quad (j = r+2, \dots, s-1),$$

and

$$(17) \quad e_re_s = c_{rs}e_n, \quad e_se_r = c_{sr}e_n.$$

Now put

$$(18) \quad e_{r+2}' = e_{r+2} + c_{rr}e_n,$$

$$(19) \quad e_j = e_j + c_{r,j-1}e_n \quad (j = r+3, \dots, s).$$

From (14) and (18) we have $e_r^2 = e_{r+2} + c_{rr}e_n = e_{r+2}'$. Then

$$\begin{aligned} e_r^3 &= e_re_{r+2}' = e_re_{r+2} = e_{r+3} + c_{r,r+2}e_n = e_{r+3}' = e_{r+2}'e_r = e_{r+2}e_r \\ &= e_{r+3} + c_{r+2,r}e_n, \text{ by (16) and (19). Hence } c_{r,r+2} = c_{r+2,r}. \end{aligned}$$

Continuing this process we have $e_r^{s-r} = e_re_r^{s-r-1} = e_re_{s-1}' =$

$$e_re_{s-1} = e_s + c_{r,s-1}e_n = e_s' = e_{s-1}'e_r = e_{s-1}e_r = e_s + c_{s-1,r}e_n$$

and $c_{r,s-1} = c_{s-1,r}$. Finally $e_r^{s-r+1} = e_re_s' = e_re_s = c_{rs}e_n = e_s'e_r = e_se_r = c_{sr}e_n$ and $c_{rs} = c_{sr}$. We now have

$$(20) \quad c_{rj} = c_{jr} \quad (j = r+2, \dots, s).$$

If $c_{rs} \neq 0$, the basal units $e_r' = e_r$, $e_n' = c_{rs}e_n$ and those defined by (18) and (19) satisfy (10) and (12). If $c_{rs} = 0$, the units $e_r' = e_r$, (18), (19) satisfy (10) and (13). It follows from (5), (18), (19) that $e_i e_j' = e_i e_j$, $e_j' e_i = e_j e_i$ ($i < r$ or $i > s$ or $i = r + 1$, $j = 1, \dots, n - 1$). Hence (18) and (19) leave (11) unaltered. If $c_{rs} \neq 0$, the transformation $e_n' = c_{rs}e_n$ merely replaces c_{ij} in (11) by $c_{ij}' = c_{rs}^{-1}c_{ij}$. Thus when $c_{rs} \neq 0$, the new algebras have the multiplication table (11), (10), (12), (5), while if $c_{rs} = 0$, they have (11), (10), (13), (5).

Using (18), (19), (15), (16), (17), (20) we find that $c_{r+p, r+q} = c_{r, p+q+r-1} = c_{p+q+r-1, r}$ for $p \geq 2$, $q \geq 2$, $p + q + r - 1 \leq s$.

Finally, we prove

THEOREM 8. Let N be a linear associative nilpotent algebra of order $n - 1$ and degree $n - r - 2$, having the multiplication table

$$e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k \quad (i < r \text{ or } i = r+1, r+3, j = 1, \dots, n-1, \text{ and } i = 1, \dots, n-1, j < r \text{ or } j = r+1, r+3).$$

$$(21) \quad e_r^2 = e_{r+2},$$

$$(22) \quad e_r^\ell = e_{r+\ell+1} \quad (\ell = 3, \dots, n - r - 2).$$

Then every algebra of order n derived from N by the use of Theorem 5 has either the multiplication table

$$(23) \quad e_i e_j = \sum_{k=1}^{n-1} c_{ijk} e_k + c_{ij} e_n \quad (i < r \text{ or } i = r+1, r+3, j = 1, \dots, n-1, \text{ and } i = 1, \dots, n-1, j < r \text{ or } j = r+1, r+3),$$

$$(24) \quad e_r^{n-r-1} = e_n,$$

with (21), (22) and (5), or the multiplication table (23), (21), (22), (5), together with

$$(25) \quad e_r^{n-r-1} = 0.$$

By Theorem 5, the multiplication tables of the new algebras are (23), (5), (14), with

$$(26) \quad e_{r+2}^2 = e_{r+5} + c_{r+2, r+2} e_n,$$

$$(27) \quad e_r e_{r+2} = e_{r+4} + c_{r, r+2} e_n, \quad e_{r+2} e_r = e_{r+4} + c_{r+2, r} e_n,$$

$$(28) \quad \begin{aligned} e_r e_j &= e_{j+1} + c_{rj} e_n, \\ e_j e_r &= e_{j+1} + c_{jr} e_n \end{aligned} \quad (j = r + 4, \dots, n - 2),$$

$$(29) \quad e_r e_{n-1} = c_{r, n-1} e_n, \quad e_{n-1} e_r = c_{n-1, r} e_n,$$

$$(30) \quad \begin{aligned} e_{r+2} e_j &= e_{j+2} + c_{r+2, j} e_n, \\ e_j e_{r+2} &= e_{j+2} + c_{j, r+2} e_n \end{aligned} \quad (j = r + 4, \dots, n - 3),$$

$$(31) \quad e_{r+2} e_j = c_{r+2, j} e_n, \quad e_j e_{r+2} = c_{j, r+2} e_n \quad (j = n - 2, n - 1),$$

$$(32) \quad \begin{aligned} e_i e_j &= e_{i+j-r-1} + c_{ij} e_n \quad (r+4 \leq i \leq n-1, r+4 \leq j \leq n-1) \\ &= c_{ij} e_n \quad \text{for } i + j - r > n. \end{aligned}$$

Now set

$$(18) \quad e_{r+2}' = e_{r+2} + c_{rr} e_n,$$

$$(33) \quad e_{r+4}' = e_{r+4} + c_{r, r+2} e_n,$$

$$(34) \quad e_j' = e_j + c_{r, j-1} e_n \quad (j = r + 5, \dots, n - 1).$$

From (14) and (18) we have $e_r^2 = e_{r+2} + c_{rr} e_n = e_{r+2}'$. Then by (27) and (33), $e_r^3 = e_r e_{r+2}' = e_r e_{r+2} = e_{r+4} + c_{r, r+2} e_n = e_{r+4}' = e_{r+2}' e_r = e_{r+2} e_r = e_{r+4} + c_{r+2, r} e_n$. Hence $c_{r, r+2} = c_{r+2, r}$. Then by (28) and (34), $e_r^4 = e_r e_{r+4}' = e_r e_{r+4} = e_{r+5} + c_{r, r+4} e_n$

$= e_{r+5}' = e_{r+4}'e_r = e_{r+4}e_r = e_{r+5} + c_{r+4,r}e_n$ and $c_{r,r+4} = c_{r+4,r}$.
 By repeated use of (28) and (34) we find $e_r^{n-r-2} = e_re_r^{n-r-3} =$
 $e_re_{n-2}' = e_re_{n-2} = e_{n-1} + c_{r,n-2}e_n = e_{n-1}' = e_r^{n-r-3}e_r = e_{n-2}'e_r$
 $= e_{n-2}e_r = e_{n-1} + c_{n-2,r}e_n$ and $c_{r,n-2} = c_{n-2,r}$. Finally by (29),
 $e_r^{n-r-1} = e_re_{n-1}' = e_re_{n-1} = c_{r,n-1}e_n = e_{n-1}'e_r = e_{n-1}e_r =$
 $c_{n-1,r}e_n$ and $c_{r,n-1} = c_{n-1,r}$. Thus we have

$$(35) \quad c_{rj} = c_{jr} \quad (j = r+2, r+4, r+5, \dots, n-1).$$

If $c_{r,n-1} \neq 0$, the basal units $e_r' = e_r$, $e_n' = c_{r,n-1}e_n$
 and those defined by (18), (33), (34) satisfy (21), (22), (24),
 while if $c_{r,n-1} = 0$, the units $e_r' = e_r$, (18), (33), (34) satisfy
 (21), (22), (25). From (18), (33), (34), (5) we see that $e_ie_j' =$
 e_ie_j , $e_j'e_i = e_je_i$ ($i < r$ or $i = r+1, r+3, j = 1, \dots, n-1$).
 Hence (18), (33), (34) leave (23) unaltered. When $c_{r,n-1} \neq 0$,
 the transformation $e_n' = c_{r,n-1}e_n$ merely replaces c_{ij} in (23) by
 $c_{ij}' = c_{r,n-1}^{-1}c_{ij}$. Hence if $c_{r,n-1} \neq 0$, the new algebras have
 the multiplication table (23), (21), (22), (24), (5). In case
 $c_{r,n-1} = 0$, the multiplication table is (23), (21), (22), (25),
 (5).

From (33), (34), (32), (28), (29), (35) we find that
 $c_{r+p,r+q} = c_{r,r+p+q-2} = c_{r+p+q-2,r}$ for $p \geq 4$, $q \geq 4$, $r+p+q-2$
 $\leq n-1$. Also from (18), (34), (30), (31), (28), (29), (35), we
 have $c_{r+2,r+q} = c_{r,r+q+1} = c_{r+q+1,r} = c_{r+q,r+2}$ for $q \geq 4$ and
 $r+q+1 \leq n-1$.

When one of the Theorems 6, 7, 8 is applied to an algebra
 of order $n-1$, the degree of e_r may be increased, that is, (4)
 or (12) or (24) may hold. In this case we shall refer to the
 result used as Theorem 6₁ or 7₁ or 8₁. If (6) or (13) or (25)
 holds we shall refer to Theorem 6₂ or 7₂ or 8₂.

CHAPTER II

ALGEBRAS OF ORDERS NOT GREATER THAN FOUR

1. Power algebras. There is no nilpotent algebra of order 1 except the zero algebra $e_1^2 = 0$. The only one of order 2 is $e_1^2 = e_2$, of degree 2, a case of the power algebra $e_1^1 = e_1$ ($i = 1, \dots, n$).

2. Algebras of order 3 not power algebras. Let A be an algebra of degree 2. Choose the basal units so that $e_2^2 = e_3$. By Theorems 2, 3, 4 the algebra A takes the form

$$e_1^2 = b_{11}e_2 + c_{11}e_3, \quad e_1e_2 = c_{12}e_3, \quad e_2e_1 = c_{21}e_3, \quad e_2^2 = e_3,$$

with unwritten products zero. By the associative law, $e_1^2e_2 = b_{11}e_2^2 + c_{11}e_3e_2 = b_{11}e_3 = c_{12}e_1e_3 = 0$. Hence $b_{11} = 0$.

First let A be commutative. Put $e_1' = e_1 - c_{12}e_2$ and obtain $e_1'e_2 = 0 = e_2e_1'$. If the new $c_{11} = 0$, the algebra is reducible and is excluded from our classification. We now have

$$e_1^2 = ae_3, \quad e_2^2 = e_3 \quad (a \neq 0),$$

which appears as algebra 2 in the summary.

When A is non-commutative, put $e_1' = \frac{1}{c_{21} - c_{12}} e_1 - \frac{c_{12}}{c_{21} - c_{12}} e_2$ and obtain $e_1'e_2 = 0$. The result is

$$e_1^2 = ae_3, \quad e_2e_1 = e_3 = e_2^2$$

which is algebra 3 in the final classification.

If the degree of A is 1 then $(e_1 + e_2)^2 = 0 = e_1e_2 + e_2e_1$ and $e_1e_2 = -e_2e_1$. Similarly $e_1e_3 = -e_3e_1$ and $e_2e_3 = -e_3e_2$.

If all products $e_1 e_j$ were zero, the algebra would be a zero algebra. Hence let $e_1 e_2 \neq 0$ and write $e_1 e_2 = a_{12} e_1 + b_{12} e_2 + c_{12} e_3$. If $c_{12} = 0$, then $0 = e_1^2 e_2 = a_{12} e_1^2 + b_{12} e_1 e_2 = b_{12} e_1 e_2$ and $b_{12} = 0$. Hence $e_1 e_2 = a_{12} e_1$. But by Theorem 1, $a_{12} = 0$ and $e_1 e_2 = 0$, contrary to our hypothesis. Hence $c_{12} \neq 0$ and we may put $e_3' = a_{12} e_1 + b_{12} e_2 + c_{12} e_3$. Then $e_3'^2 = 0$, $e_1 e_2 = e_3'$, $e_1^2 e_2 = 0 = e_1 e_3' = -e_3' e_1$. Also $e_1 e_2^2 = 0 = e_3' e_2 = -e_2 e_3'$, and A takes the form

$$e_1 e_2 = -e_2 e_1 = e_3$$

which we call algebra 4.

3. Algebras of order 4 not power algebras. The algebras of degree 3 are derived from algebras 2 or 3 in 3 basal units or from the reducible algebra $e_2^2 = e_3$, by the use of Theorem 6₁ with $s = n - 1$. The new algebras take the form

$$(36) \quad \begin{aligned} e_1^2 &= c_{11} e_3 + d_{11} e_4, & e_1 e_2 &= d_{12} e_4, & e_2 e_1 &= c_{21} e_3 + d_{21} e_4, \\ e_1 e_3 &= d_{13} e_4, & e_3 e_1 &= d_{31} e_4, & e_2^2 &= e_3, & e_2^3 &= e_4. \end{aligned}$$

But $e_1 e_2^2 = 0 = e_1 e_3$, $e_2 e_1 e_2 = 0 = c_{21} e_3 e_2 = c_{21} e_4$ and hence $c_{21} = 0$. Then $e_2^2 e_1 = 0 = e_3 e_1$. Also $e_1^2 e_2 = 0 = c_{11} e_3 e_2 = c_{11} e_4$ and $c_{11} = 0$.

If the algebra (36) is commutative, the transformation $e_1' = e_1 - d_{12} e_3$ gives

$$(37) \quad e_1'^2 = a e_4, \quad e_2'^2 = e_3, \quad e_2'^3 = e_4 \quad (a \neq 0),$$

or the reducible algebra $e_2'^2 = e_3$, $e_2'^3 = e_4$ in case $a = 0$. We call (37) algebra 2 in the summary.

If (36) is non-commutative, put $e_1' = \frac{1}{d_{21} - d_{12}} e_1 - \frac{d_{12}}{d_{21} - d_{12}} e_3$ and obtain

$$e_1^2 = ae_4, \quad e_2e_1 = e_4, \quad e_2^2 = e_3, \quad e_2^3 = e_4$$

which is algebra 3.

Now let A be an algebra of degree 2 arising from the zero algebra of order 3 by the use of Theorem 5. Then $A = (e_1, \dots, e_4)$ where $e_1e_j = d_{1j}e_4$ (d_{1j} in F , $i, j = 1, \dots, 4$). Since A has degree 2, we may, without loss of generality, take $e_3^2 \neq 0$, replace $c_{33}e_4$ by e_4 and have A in the form

$$(38) \quad e_1e_j = d_{1j}e_4 \quad (i, j = 1, \dots, 4; d_{33} = 1).$$

Put $e_1' = e_1 - d_{13}e_3$, $e_2' = e_2 - d_{23}e_3$. Then $e_1'e_3 = e_2'e_3 = 0$. Hence we may assume that

$$(39) \quad e_1e_3 = e_2e_3 = 0.$$

If A is commutative then $e_3e_1 = e_3e_2 = 0$ and if also $e_1^2 = e_2^2 = e_1e_2 = 0$ clearly A is reducible. Hence one of d_{11} , d_{22} , $d_{12} = d_{21}$ is not zero and there exist ℓ, m such that $a = \ell^2d_{11} + 2\ell md_{12} + m^2d_{22} \neq 0$. But $e_1' = \ell e_1 + me_2$ gives $e_1'^2 = ae_4$, leaves (39) unaltered and we may assume that $e_1^2 = ae_4 \neq 0$. Now $e_1(re_1 + e_2) = (ra + d_{12})e_4 = 0$ for $r = -d_{12}a^{-1}$. Put $e_2' = re_1 + e_2$ and obtain $e_1e_2' = 0$. Since A is commutative it has the multiplication table given by

$$(40) \quad e_1^2 = ae_4, \quad e_2^2 = be_4, \quad e_3^2 = e_4 \quad (ab \neq 0).$$

The condition $ab \neq 0$ is clearly necessary in order that A be irreducible. Hence (40) is algebra 4 in the final classification.

Next let A be non-commutative. If $e_3e_2 \neq 0$, put $e_1' = e_1 - d_{32}^{-1}d_{31}e_2$ and obtain

$$(41) \quad e_3e_1 = 0.$$

If $e_3e_2 = 0$, $e_3e_1 \neq 0$, the interchange of e_1 and e_2 gives (41) and $e_3e_2 \neq 0$. If $e_3e_2 = 0 = e_3e_1$, then since A is not commutative, one of e_1e_2 and e_2e_1 is not zero and there is no loss of generality if we assume that $e_2e_1 \neq 0$. Then also $d_{21} \neq d_{12}$. Choose $d \neq 0$ such that $d_{11} + (dd_{21})^2 \neq 0$. This is always possible since $d_{21} \neq 0$. Take

$$e_1' = dd_{21}e_1 + d_{11}e_3, \quad e_2' = e_2 + d^{-1}e_3, \quad e_3' = e_1 - dd_{21}e_3.$$

Then $e_1'e_3' = e_3'e_1' = 0 = e_2'e_3'$, $e_3'e_2' = (d_{12} - d_{21})e_4 \neq 0$, $e_3'^2 = [d_{11} + (dd_{21})^2]e_4 \neq 0$. Replacing e_4 by $e_3'^2$ we have a new multiplication table with $e_3e_2 \neq 0$. Replacing e_2 by $d_{32}^{-1}e_2$ we have (38), (39), (41) and $e_3e_2 = e_4$.

Suppose that $e_1e_2 = 0$. If $e_1^2 \neq 0$ then $e_2' = d_{11}^{-1}e_1 + e_2$ yields $e_1e_2' = e_4$, while $e_2'e_3 = 0$, $e_3e_2' = e_4$. If $e_1^2 = 0$, then $e_2e_1 \neq 0$ since otherwise A is reducible. Take

$$e_1' = d_{21}^{-1}e_1 + e_3, \quad e_2' = e_3, \quad e_3' = (d_{22} - 1)d_{21}^{-1}e_1 - e_2 + e_3.$$

Then $e_1'e_2' = e_4$, $e_1'e_3' = e_3'e_1' = 0 = e_2'e_3'$, $e_3'e_2' = e_4 = e_3'^2$, and we have shown that every non-commutative algebra of degree 2 arising from the zero algebra of order 3 has the form

$$e_1^2 = ae_4, \quad e_2^2 = be_4, \quad e_1e_2 = e_3e_2 = e_4 = e_3^2, \quad e_2e_1 = ce_4,$$

which is algebra 5 in the summary.

We next consider algebras arising from algebras 2 and 3 in 3 basal units and from the reducible algebra $e_2^2 = e_3$ by the use of Theorem 6_a with $s = n - 1$. These also are of degree 2 and take the form

$$(42) \quad \begin{aligned} e_1^2 &= c_{11}e_3 + d_{11}e_4, & e_2^2 &= e_3, & e_2e_1 &= c_{21}e_3 + d_{21}e_4, \\ e_1e_2 &= d_{12}e_4, & e_1e_3 &= d_{13}e_4, & e_3e_1 &= d_{31}e_4. \end{aligned}$$

But $e_1 e_2^2 = 0 = e_1 e_3$, $e_2^2 e_1 = 0 = e_3 e_1$, by the associative law.

First let (42) be commutative. Then $c_{21} = 0$. If $d_{12} = d_{11} = 0$, the algebra is reducible. If $d_{12} = 0$, $d_{11} \neq 0$, put $e_4' = c_{11} e_3 + d_{11} e_4$ and obtain the reducible algebra $e_1^2 = e_4$, $e_2^2 = e_3$. Hence assume that $d_{12} \neq 0$ and take $e_1' = d_{12}^{-1} e_1$, obtaining $e_1' e_2 = e_4 = e_2 e_1'$. If now $c_{11} = 0$, $d_{11} \neq 0$, put $e_2' = e_1 - d_{11} e_2$, $e_3' = d_{11}^2 e_3 - d_{11} e_4$ and obtain $e_1^2 = d_{11} e_4$, $e_2^2 = e_3$, which is reducible. If $c_{11} \neq 0$, $d_{11} \neq 0$, let $b = c_{11} + \frac{1}{4} d_{11}^2$. If $b \neq 0$, the transformation $e_1' = b e_2$, $e_2' = e_1 - \frac{1}{2} d_{11} e_2$, $e_3' = b e_3$, $e_4' = -\frac{1}{2} d_{11} b e_3 + b e_4$ gives the algebra

$$(43) \quad e_1^2 = b e_3, \quad e_1 e_2 = e_4 = e_2 e_1, \quad e_2^2 = e_3.$$

If $b = 0$, put $e_1' = e_1 - \frac{1}{2} d_{11} e_2$, $e_2' = 2e_1 - \frac{1}{2} d_{11} e_2$, $e_3' = -\frac{3}{4} d_{11}^2 e_3 + 2d_{11} e_4$, $e_4' = -\frac{1}{4} d_{11}^2 e_3 + \frac{1}{2} d_{11} e_4$ and again obtain (43).

If $b = k^2$, k in F , the transformation $e_1' = e_1 - k e_2$, $e_2' = e_1 + k e_2$, $e_3' = 2k^2 e_3 + 2k e_4$, $e_4' = 2k^2 e_3 - 2k e_4$ reduces (43) to $e_1^2 = e_4$, $e_2^2 = e_3$ which is reducible. Hence we assume that b is not equal to the square of a quantity in F . Interchanging e_3 and e_4 we now have

$$e_1^2 = b e_4, \quad e_1 e_2 = e_3 = e_2 e_1, \quad e_2^2 = e_4 \quad (b \neq k^2, k \text{ in } F),$$

which is algebra 6 in the summary.

Next let algebra (42) be non-commutative. Suppose that $d_{12} = 0 = d_{21}$. Then d_{11} and c_{21} must both be different from zero; for if $d_{12} = d_{21} = d_{11} = 0$, the algebra is reducible, while if $d_{12} = d_{21} = c_{21} = 0$, it is commutative. Under the transformation

$$\begin{aligned}
 e_1' &= \frac{1}{2c_{21}} e_1 + e_2, & e_2' &= -\frac{1}{2c_{21}} e_1 + e_2, \\
 e_3' &= \left[\frac{1}{2} + \frac{c_{11}}{4c_{21}^2} \right] e_3 + \frac{d_{11}}{4c_{21}^2} e_4, & e_4' &= \left[\frac{1}{2} - \frac{c_{11}}{4c_{21}^2} \right] e_3 - \frac{d_{11}}{4c_{21}^2} e_4,
 \end{aligned}$$

the algebra becomes

$$(44) \quad e_1^2 = 2e_3 + e_4, \quad e_1e_2 = e_4, \quad e_2e_1 = e_3 + 2e_4, \quad e_2^2 = e_3.$$

If $d_{12} = 0$, $d_{21} \neq 0$, put $e_1' = e_1 + e_2$, $e_4' = (1 + c_{21})e_3 + d_{21}e_4$, and obtain

$$(45) \quad e_1^2 = fe_3 + ge_4, \quad e_1e_2 = e_3, \quad e_2e_1 = e_4, \quad e_2^2 = e_3.$$

If $d_{12} \neq 0$, $c_{21} = 0$, then $d_{12} \neq d_{21}$ since the algebra (42) is now non-commutative. Put $e_1' = \frac{1}{d_{12}} e_1 + \frac{d_{12}}{d_{12} - d_{21}} e_2$, $e_4' = \frac{d_{12}}{d_{12} - d_{21}} e_3 + e_4$, and obtain

$$(46) \quad e_1^2 = f'e_3 + g'e_4, \quad e_1e_2 = e_4, \quad e_2e_1 = e_3 + he_4, \quad e_2^2 = e_3.$$

If $d_{12} \neq 0$, $c_{21} \neq 0$ in (42) we may obviously obtain (46) by putting $e_1' = c_{21}^{-1}e_1$, $e_4' = d_{12}c_{21}^{-1}e_4$.

Algebras (44) and (46) are included in

$$(47) \quad e_1^2 = be_3 + ae_4, \quad e_1e_2 = e_4, \quad e_2e_1 = e_3 + de_4, \quad e_2^2 = e_3,$$

while (45) is a case of the reciprocal of (47). Now interchange e_3 and e_4 in (47) and have $e_1^2 = ae_3 + be_4$, $e_1e_2 = e_3$, $e_2e_1 = de_3 + e_4$, $e_2^2 = e_4$ which is algebra 7 in the final classification.

We next consider algebras of degree 2 arising from algebra 4 in 3 basal units by the use of Theorem 5. These algebras take the form

$$\begin{aligned}
 e_1^2 &= d_{11}e_4 & (i = 1, 2, 3), \\
 (48) \quad e_1e_2 &= e_3 + d_{12}e_4, & e_2e_1 &= -e_3 + d_{21}e_4, \\
 e_1e_3 &= d_{13}e_4, & e_3e_1 &= d_{31}e_4, \quad e_2e_3 = d_{23}e_4, \quad e_3e_2 = d_{32}e_4.
 \end{aligned}$$

But $e_1 e_2 e_3 = 0 = e_3^2$, $e_1^2 e_2 = 0 = e_1 e_3$, $e_2 e_1^2 = 0 = -e_3 e_1$,
 $e_1 e_2^2 = 0 = e_3 e_2$, $e_2^2 e_1 = 0 = -e_2 e_3$. Without loss of generality
 we may take $e_2^2 \neq 0$, put $e_4' = d_{22} e_4$ and write (48) in the form

$$(49) \quad e_1^2 = d_{11} e_4, \quad e_2^2 = e_4, \quad e_1 e_2 = e_3 + d_{12} e_4, \quad e_2 e_1 = -e_3 + d_{21} e_4.$$

If $d_{12} = -d_{21}$, put $e_1' = e_1 + \frac{1}{2} e_2$, $e_3' = e_3 + (d_{12} + \frac{1}{2}) e_4$
 and obtain $e_1'^2 = (d_{11} + \frac{1}{4}) e_4$, $e_2'^2 = e_4$, $e_1' e_2' = e_3$, $e_2' e_1' = -e_3 + e_4$,
 a case of algebra 7.

If $d_{12} \neq -d_{21}$, put $e_2' = d e_2$, $e_3' = d e_3 + d_{12} d e_4$,
 $e_4' = d^2 e_4$, where $d = d_{12} + d_{21}$. Again (49) takes the form of
 algebra 7.

There are two possible sources of algebras of degree one.
 We first consider those derived from algebra 4 in 3 basal units
 by Theorem 5. They take the form

$$(50) \quad \begin{aligned} e_1 e_2 &= e_3 + d_{12} e_4 = -e_2 e_1, \\ e_1 e_3 &= d_{13} e_4 = -e_3 e_1, & e_2 e_3 &= d_{23} e_4 = -e_3 e_2. \end{aligned}$$

The associative law gives $e_1^2 e_2 = 0 = e_1 e_3$, $e_1 e_2^2 = 0 = e_3 e_2$.
 If we put $e_3' = e_3 + d_{12} e_4$, (50) becomes $e_1 e_2 = e_3 = -e_2 e_1$,
 a reducible algebra.

Next we consider algebras arising from the zero algebra
 in 3 basal units by the use of Theorem 5. The multiplication
 table is

$$\begin{aligned} e_1 e_2 &= d_{12} e_4 = -e_2 e_1, \\ e_1 e_3 &= d_{13} e_4 = -e_3 e_1, & e_2 e_3 &= d_{23} e_4 = -e_3 e_2. \end{aligned}$$

If all $d_{ij} = 0$, we have a zero algebra. Hence we assume $d_{12} \neq 0$
 and put $e_3' = d_{12}^{-1} d_{23} e_1 - d_{12}^{-1} d_{13} e_2 + e_3$, $e_4' = d_{12} e_4$, obtain-
 ing $e_1 e_3 = 0 = e_2 e_3$. The algebra is now $e_1 e_2 = e_4 = -e_2 e_1$, which
 is reducible.

We have shown that no irreducible algebras of degree 1
and order 4 exist.

CHAPTER III

REGULAR ALGEBRAS

1. The concept of regular algebra. It will be observed that all of the algebras of orders 3 and 4 except algebra 4 of order 3 and algebras 6 and 7 of order 4 are of the type described in the following

DEFINITION. A regular nilpotent algebra of order n and degree $n - r + 1$ is one having a basal unit $e = e_r$ of this highest degree ≥ 2 such that

$$e^2 = e_{r+1}, \quad e^3 = e_{r+2}, \quad \dots, \quad e^{n-r+1} = e_n$$

and if either $1 < r$ or $j < r$ then

$$e_i e_j = c_{ij} e_n.$$

We shall in fact prove that all regular algebras have the form

$$(51) \quad \begin{aligned} e_i e_j &= c_{ij} e_n & (i, j = 1, \dots, r-1), \\ e e_{r-1} &= c e_n, & e^{1-r+1} = e_1 & (i = r, \dots, n). \end{aligned}$$

Observe that the algebras of orders 3 and 4 described above are already in the form (51) which includes the commutative algebras

$$(52) \quad \begin{aligned} e_i^2 &= c_i e_n & (i = 1, \dots, r-1), \\ e^{1-r+1} &= e_1 & (i = r, \dots, n) \end{aligned}$$

and the non-commutative algebras

$$(53) \quad \begin{aligned} e_i e_j &= c_{ij} e_n & (i, j = 1, \dots, r-1), \\ e e_{r-1} &= e_n, & e^{1-r+1} = e_1 & (i = r, \dots, n). \end{aligned}$$

2. Algebras of degree n . Every algebra of degree n is regular, with $r = 1$. It is the power algebra $e_1^1 = e_1$ ($i = 1, \dots, n$). Every algebra of order n derived from the power algebra in $n - 1$ basal units by Theorem 6₁ is evidently a power algebra.

3. Algebras of degree $n - 1$. Clearly, an algebra of degree $n - 1$ cannot be derived from the power algebra of order $n - 1$. Hence it must arise from an algebra of degree $n' - 1$ of order $n' = n - 1$ by the use of Theorem 6₁. The algebras of degree $n - 1$ of orders 3 and 4 are regular. Hence by the induction to be made in the next section, we shall see that all algebras of degree $n - 1$ are regular. They are algebras 2 and 3 in the classification of algebras of order n .

4. The structure of a regular algebra. Evidently a regular algebra of order n must be derivable from a regular algebra of order $n - 1$ by the use of Theorem 6₁ with $s = n - 1$, or from the zero algebra of order $n - 1$ by Theorem 5. It may happen, as we shall see later, that a regular algebra arises from an irregular one, but such an algebra must also be derivable from a regular one by the use of Theorem 6₁ or from a zero algebra by use of Theorem 5. Hence, to prove by induction on the order n that all regular algebras are of form (52) or (53), it is sufficient to show that all algebras in n basal units derived from (52) and (53) in $n - 1$ units by Theorem 6₁, or from the zero algebra in $n - 1$ units by Theorem 5 have the form (52) or (53).

Since (51) includes (52) and (53) and the zero algebra of order n , consider the algebras arising from (51) in $n - 1$ units. They take the form

$$\begin{aligned}
 e_1 e_j &= b_{1j} e_{n-1} + c_{1j} e_n & (i, j = 1, \dots, r-1), \\
 e e_{r-1} &= b_{r, r-1} e_{n-1} + c_{r, r-1} e_n, \\
 e_1 e_j &= c_{1j} e_n & (i = 1, \dots, r-1; j = r, \dots, n-1 \text{ and} \\
 (54) \quad & i = r+1, \dots, n-1; j = 1, \dots, r-1), \\
 e e_j &= c_{rj} e_n & (j = 1, \dots, r-2), \\
 e^{i-r+1} &= e_i & (i = r, \dots, n).
 \end{aligned}$$

By the associative law $ee_{r-1}e = 0 = b_{r, r-1}e_{n-1}e = b_{r, r-1}e_n$.

Hence $b_{r, r-1} = 0$. Also $e_1 e^2 = 0 = e_1 e_{r+1}$, $e_1 e^3 = 0 = e_1 e_{r+2}$, ..., $e_1 e^{n-r} = 0 = e_1 e_{n-1}$, $e^2 e_1 = 0 = e_{r+1} e_1$, ..., $e^{n-r} e_1 = 0 = e_{n-1} e_1$ ($i = 1, \dots, r-1$), and $e_1 e_j e = 0 = b_{1j} e_{n-1} e = b_{1j} e_n$ ($i, j = 1, \dots, r-1$). Hence $b_{1j} = 0$ ($i, j = 1, \dots, r-1$) and we may write (54) in the form

$$\begin{aligned}
 e_1 e_j &= c_{1j} e_n, \quad e_1 e = c_{1r} e_n, \quad e e_1 = c_{r1} e_n & (i, j = 1, \dots, r-1), \\
 (55) \quad & e^{i-r+1} = e_i & (i = r, \dots, n).
 \end{aligned}$$

Now put $e_1' = e_1 - c_{1r} e_{n-1}$ ($i = 1, \dots, r-1$) and obtain

$$(56) \quad e_1 e = 0 \quad (i = 1, \dots, r-1).$$

If now the c_{r1} are not all zero, we may assume, without loss of generality, that $c_{r, r-1} \neq 0$ and put $e_1' = e_1 - c_{r, r-1}^{-1} c_{r1} e_{r-1}$ ($i = 1, \dots, r-2$). This transformation leaves (56) unaltered and gives

$$(57) \quad e e_1 = 0 \quad (i = 1, \dots, r-2).$$

We have therefore proved that all regular algebras have the form¹ (51).

¹The fact that the b_{1j} and $b_{r, r-1}$ in (54) must be zero shows that the regular algebra in n units actually arises from the reducible algebra $e^{i-r+1} = e_i$ ($i = r, \dots, n-1$). Similar cases will arise in irregular algebras also.

5. Commutative regular algebras. We now write (51) in the form

$$(58) \quad \begin{aligned} e_1 e_j &= c_{1j} e_n & (i, j = 1, \dots, r-1; c_{1j} = c_{j1}), \\ e_1^{i-r+1} &= e_1 & (i = r, \dots, n). \end{aligned}$$

Suppose that $e_1 e_2 \neq 0$. If also $e_1^2 \neq 0$, put $e_2' = -c_{11}^{-1} c_{12} e_1 + e_2$ and obtain

$$(59) \quad e_1 e_2 = 0 = e_2 e_1.$$

If $e_1^2 = 0$, $e_2^2 \neq 0$, then the transformation $e_1' = e_1 - c_{22}^{-1} c_{12} e_2$ gives (59), while if $e_1^2 = 0 = e_2^2$, we put $e_1' = e_1 + e_2$, $e_2' = e_1 - e_2$ and obtain (59).

Now suppose that $e_1 e_3 \neq 0$. If $e_1^2 \neq 0$, put $e_3' = -c_{11}^{-1} c_{13} e_1 + e_3$. If $e_1^2 = 0$, $e_3^2 \neq 0$, put $e_1' = e_1 - c_{33}^{-1} c_{13} e_3$, $e_2' = -c_{13}^{-1} c_{23} e_1 + e_2$. If $e_1^2 = 0 = e_3^2$, put $e_1' = e_1 + e_3$, $e_2' = -c_{13}^{-1} c_{23} e_1 + e_2$, $e_3' = e_1 - e_3$. In each of these three cases (59) is unaltered and we obtain

$$(60) \quad e_1 e_3 = 0 = e_3 e_1.$$

We next suppose that $e_1 e_4 \neq 0$. Using transformations similar to those above, we obtain $e_1 e_4 = 0 = e_4 e_1$ and leave (59) and (60) unaltered.

Continuing this step-by-step process of eliminating products containing e_1 , we arrive at a form of (58) in which

$$(61) \quad e_1 e_i = 0 = e_i e_1 \quad (i = 2, \dots, r-2).$$

Suppose that $c_{1,r-1} \neq 0$. If also $e_1^2 \neq 0$, put $e_{r-1}' = -c_{11}^{-1} c_{1,r-1} e_1 + e_{r-1}$ and obtain

$$(62) \quad e_1 e_{r-1} = 0 = e_{r-1} e_1.$$

If $e_1^2 = 0$, $e_{r-1}^2 \neq 0$, put $e_1' = e_1 - c_{r-1,r-1}^{-1} c_{1,r-1} e_{r-1}$,
 $e_1' = -c_{1,r-1}^{-1} c_{1,r-1} e_1 + e_1$ ($i = 2, \dots, r-2$), and obtain (62).
 If $e_1^2 = 0 = e_{r-1}^2$, the transformation $e_1' = e_1 + e_{r-1}$,
 $e_1' = -c_{1,r-1}^{-1} c_{1,r-1} e_1 + e_1$ ($i = 2, \dots, r-2$), $e_{r-1}' = e_1 - e_{r-1}$
 gives (62). In each of these three cases (61) is left unaltered.
 We now have

$$(63) \quad e_1 e_i = 0 = e_i e_1 \quad (i = 2, \dots, r-1).$$

To eliminate the products $e_2 e_i$ ($i = 3, \dots, r-1$) we first interchange e_1 and e_2 obtaining

$$(64) \quad e_2 e_1 = 0 = e_1 e_2 \quad (i = 1, 3, 4, \dots, r-1).$$

We then use the above transformations to eliminate the products $e_1 e_i$ ($i = 3, \dots, r-1$). These transformations leave (64) unaltered and we now have (58) together with (63) and (64).

We next obtain $e_3 e_1 = 0 = e_1 e_3$ ($i = 4, \dots, r-1$) by the same method, and continuing this process we finally have (52). It is evident that we must have $c_i \neq 0$ ($i = 1, \dots, r-1$). Otherwise the algebra is reducible.

6. Non-commutative regular algebras. We shall now show that if (51) is not commutative it can be written in the form (53). If $c \neq 0$, we merely put $e_{r-1}' = c^{-1} e_{r-1}$ and have (53).

Now suppose that $c = 0$. Then some $c_{pq} \neq c_{qp}$ ($p \leq r-1$, $p \neq q$). Since both cannot be zero, we may assume that $c_{pq} \neq 0$, $p < q$. If $q \neq r-1$, interchange e_q and e_{r-1} . Then $p < r-1$, $c_{p,r-1} \neq c_{r-1,p}$ and $c_{p,r-1} \neq 0$. Also $d = c_{p,r-1} - c_{r-1,p} \neq 0$.

First suppose that $c_{r-1,r-1} \neq 0$. If $n > r+1$, put $e_{r-1}' = -d^{-1} e_p + c_{r-1,r-1}^{-1} d^{-1} c_{p,r-1} e_{r-1}$, $e_r' = e_{r-1} + e_r$,

$e_{r+1}' = e_{r+1} + c_{r-1,r-1}e_n$. Algebras (51) become

$$(65) \quad \begin{aligned} e_i e_j &= c_{ij} e_n & (i, j = 1, \dots, r-1), \\ e_i e &= c_{ir} e_n, & e e_i = c_{ri} e_n & (i = 1, \dots, r-2), \\ e e_{r-1} &= e_n, & e^{i-r+1} = e_i & (i = r, \dots, n). \end{aligned}$$

If $n = r + 1$, we employ the transformation

$$\begin{aligned} e_{r-1}' &= -g d^{-1} e_p + g c_{r-1,r-1}^{-1} c_{p,r-1} d^{-1} e_{r-1}, \\ e_r' &= h e_{r-1} + e_r, & e_n' &= k e_n \end{aligned}$$

For $c_{r-1,r-1} \neq -1$, put $g = c_{r-1,r-1} + 1 = k$, $h = 1$. If $c_{r-1,r-1} = -1$, use $g = -3/2$, $h = 2$, $k = -3$. In either case, our algebras (51) take the form (65).

Now suppose that $c_{r-1,r-1} = 0$. Then (51) becomes (65) under the transformation

$$e_{r-1}' = -d^{-1} e_p + e_{r-1} + d^{-1} c_{p,r-1} e_{n-1}, \quad e_r' = e_{r-1} + e_r.$$

To reduce (65) to (53), first put $e_i' = e_i - c_{ir} e_{n-1}$ ($i = 1, \dots, r-2$) and obtain (56). Then put $e_i' = e_i - c_{ri} e_{r-1}$ ($i = 1, \dots, r-2$). This transformation leaves (56) unaltered and gives (57). Thus (65) takes the form (53).

We have shown that the multiplication table of any regular algebra may be written in the form (52) or (53).

CHAPTER IV

ALGEBRAS OF DEGREE $n - 2$ ($n > 4$)

1. Algebras arising by the use of Theorem 6₁ with $s = n - 1$ from algebras 4 and 5 and from reducible algebras of the same type in $n - 1$ basal units. Since algebras 4 and 5 in 4 units are regular, the induction of Chapter III shows that the algebras considered in this section are regular, and hence in the commutative case we have algebra 4 of order n in the final classification. The non-commutative algebra is

$$(66) \quad \begin{aligned} e_1^2 &= c_{11}e_n, & e_2^2 &= c_{22}e_n, & e_1e_2 &= c_{12}e_n, & e_2e_1 &= c_{21}e_n, \\ e_3e_2 &= e_n, & e_3^{i-2} &= e_1 & & & (i = 3, \dots, n). \end{aligned}$$

To put (66) into the form given in the summary, we proceed as follows.

If $e_1e_2 \neq 0$, put $e_1' = c_{12}^{-1}e_1$ and obtain algebra 5. Now let $e_1e_2 = 0$, $e_2e_1 \neq 0$. Put $e_1' = ae_1 + be_2 + ce_n$, $e_2' = de_1$, $e_3' = ge_2 + e_3$, $e_4' = e_4 + he_n$. If $e_1^2 \neq 0$, use $a = c_{21}^{-1}$, $b = -\frac{1}{2}c_{21}^{-2}c_{11}$, $c = \frac{1}{4}c_{21}^{-4}c_{11}^2c_{22}$, $d = 2c_{11}^{-1}c_{21}$, $g = \frac{1}{2}c_{21}^{-2}c_{11}$, $h = \frac{1}{2}c_{21}^{-2}c_{11}(\frac{1}{2}c_{21}^{-2}c_{11}c_{22} + 1)$. If $e_1^2 = 0$, put $a = -c_{21}^{-1}$, $b = 1$, $c = -c_{22}$, $d = c_{21}^{-1}$, $g = 1$, $h = c_{22} + 1$. The result is algebra 5. Now suppose that $e_1e_2 = 0 = e_2e_1$. If also $c_{11} = 0$, the algebra is reducible. Hence let $c_{11} \neq 0$. Put $e_2' = c_{11}^{-1}e_1 + e_2$ and obtain algebra 5.

2. Algebras derived from algebras 2 and 3 in $n - 1$ units by means of Theorem 6₂ with¹ $s = n - 1$. The new algebras take

¹The algebras (67) include also those derived from

the form

$$(67) \quad \begin{aligned} e_1^2 &= a_{11}e_{n-1} + b_{11}e_n, & e_1e_2 &= b_{12}e_n, & e_2e_1 &= a_{21}e_{n-1} + b_{21}e_n, \\ e_1e_i &= b_{1i}e_n, & e_ie_1 &= b_{i1}e_n & (i &= 3, \dots, n-1), \\ e_2^{i-1} &= e_1 & (i &= 2, \dots, n-1). \end{aligned}$$

By the associative law we have $e_1e_2^2 = 0 = e_1e_3$, $e_2^2e_1 = a_{21}e_2e_{n-1} = a_{21}e_2^{n-1} = 0 = e_3e_1$. Hence $e_1e_2^3 = 0 = e_1e_4$, $e_1e_2^4 = 0 = e_1e_5$, ..., $e_1e_2^{n-2} = 0 = e_1e_{n-1}$, and $e_2^3e_1 = 0 = e_4e_1$, ..., $e_2^{n-2}e_1 = 0 = e_{n-1}e_1$.

First let (67) be commutative. Then $a_{21} = 0$ and $b_{12} = b_{21}$. If $b_{12} = b_{11} = 0$, the algebra is reducible. In case $b_{12} = 0$, $b_{11} \neq 0$, put $e_n' = a_{11}e_{n-1} + b_{11}e_n$. The algebra becomes $e_1^2 = e_n$, $e_2^{i-1} = e_1$ ($i = 2, \dots, n-1$), which is reducible. Hence let $b_{12} \neq 0$. Now put $e_1' = b_{12}^{-1}e_1$ and obtain $e_1e_2 = e_n = e_2e_1$. If now $a_{11} = 0$, $b_{11} \neq 0$, the transformation $e_2' = e_1 - b_{11}e_2$, $e_3' = b_{11}^2e_3 - b_{11}e_n$, $e_i' = (-b_{11})^{i-1}e_i$ ($i = 4, \dots, n-1$) reduces the algebra to $e_1^2 = b_{11}e_n$, $e_2^{i-1} = e_1$ ($i = 2, \dots, n-1$), which is reducible. We therefore must have $a_{11} \neq 0$ or $a_{11} = b_{11} = 0$ if the algebra is to be irreducible. We now have the commutative algebra

$$(68) \quad \begin{aligned} e_1^2 &= be_{n-1} + ae_n & (b \neq 0 \text{ or } a = b = 0), \\ e_1e_2 &= e_n = e_2e_1, & e_2^{i-1} &= e_1 \quad (i = 2, \dots, n-1). \end{aligned}$$

Next let (67) be non-commutative. If $a_{21} \neq 0$, we may put $e_1' = a_{21}^{-1}e_1$ and write (67) in the form

$$(69) \quad \begin{aligned} e_1^2 &= be_{n-1} + ae_n, & e_1e_2 &= ce_n, & e_2e_1 &= e_{n-1} + de_n, \\ e_2^{i-1} &= e_1 & (i &= 2, \dots, n-1). \end{aligned}$$

reducible algebras of the same type as algebras 2 and 3. Hereafter in such cases, we shall not mention the irreducible algebras, since the form of the new algebras makes it clear that those from which they are derived may be reducible.

If $a_{21} = 0$, then $b_{12} \neq b_{21}$ and one of them must be different from zero. Without loss of generality, we may assume that $b_{12} \neq 0$ and put

$$e_1' = \frac{b_{12}}{b_{12} - b_{21}} e_1 + \frac{b_{12}}{b_{12} - b_{21}} e_{n-2}, \quad e_n' = b_{12}^{-1} e_{n-1} + e_n,$$

again obtaining (69).

We observe that for algebras (68) and (69) the linear sets B_i are $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_n)$, $B_3 = (e_4)$, $B_4 = (e_5)$, ..., $B_{n-2} = (e_{n-1})$. If we subject the basal units to the cyclic permutation $(e_3 e_4 \dots e_{n-1} e_n)$, the algebras (68) and (69) become

$$(70) \quad \begin{aligned} e_1^2 &= a e_3 + b e_n & (b \neq 0 \text{ or } a = b = 0), \\ e_1 e_2 &= e_3 = e_2 e_1, & e_2^{1-2} = e_1 \quad (i = 4, \dots, n) \end{aligned}$$

and

$$(71) \quad \begin{aligned} e_1^2 &= a e_3 + b e_n, & e_1 e_2 &= c e_3, & e_2 e_1 &= d e_3 + e_n, \\ & & e_2^{1-2} &= e_1 & & (i = 4, \dots, n). \end{aligned}$$

respectively, and the B_i are now $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_4)$, $B_3 = (e_5)$, $B_4 = (e_6)$, ..., $B_{n-2} = (e_n)$.

3. Algebras derived from algebras (70) and (71) in $n - 1$ units by means of Theorem 7₁ with $s = n - 1$. The new algebras have the form

$$(72) \quad \begin{aligned} e_1^2 &= a_{11} e_3 + b_{11} e_{n-1} + c_{11} e_n, & e_1 e_2 &= a_{12} e_3 + c_{12} e_n, \\ e_2 e_1 &= a_{21} e_3 + b_{21} e_{n-1} + c_{21} e_n, & & \\ e_1 e_i &= c_{1i} e_n, & e_i e_1 &= c_{i1} e_n & (i = 3, \dots, n - 1), \\ e_3 e_1 &= c_{31} e_n, & e_1 e_3 &= c_{13} e_n & (i = 2, \dots, n - 1), \\ e_2^{1-2} &= e_1 & & & (i = 4, \dots, n). \end{aligned}$$

By the associative law $e_1 e_4 e_2 = 0 = e_1 e_2^3 = e_1 e_5$, $e_2 e_4 e_1 = 0 = e_2^3 e_1 = e_5 e_1$ and hence $e_1 e_2^4 = 0 = e_1 e_6$, $e_1 e_2^5 = 0 = e_1 e_7$, ..., $e_1 e_2^{n-3} = 0 = e_1 e_{n-1}$ and $e_2^4 e_1 = 0 = e_6 e_1$, ..., $e_2^{n-3} e_1 = 0 = e_{n-1} e_1$.

Therefore we have

$$(73) \quad e_1 e_i = 0 = e_i e_1 \quad (i = 5, \dots, n-1).$$

In a similar manner we obtain

$$(74) \quad e_3 e_i = 0 = e_i e_3 \quad (i = 4, \dots, n-1).$$

Also $e_1 e_2^2 = a_{12} e_3 e_2 = a_{12} c_{32} e_n = e_1 e_4 = c_{14} e_n$. Hence $c_{14} = a_{12} c_{32}$. If $n > 5$, set $e_1' = e_1 - c_{14} e_{n-2} - c_{12} e_{n-1}$, $e_3' = e_3 - c_{32} e_{n-1}$. If $n = 5$, put $e_1' = e_1 - c_{14} e_2 - c_{12} e_4$, $e_3' = e_3 - c_{32} e_4$. In either case we have $e_1' e_2 = a_{12} e_3'$, $e_1' e_4 = 0 = e_3' e_2$ and we may now write (72) in the form

$$(75) \quad \begin{aligned} e_1'^2 &= a_{11} e_3' + b_{11} e_{n-1} + c_{11} e_n, & e_1' e_2 &= a_{12} e_3', \\ e_2' e_1 &= a_{21} e_3' + b_{21} e_{n-1} + c_{21} e_n, & e_1' e_3 &= c_{13} e_n, \\ e_3' e_1 &= c_{31} e_n, & e_4 e_1 &= c_{41} e_n, & e_2 e_3 &= c_{23} e_n, & e_3'^2 &= c_{33} e_n, \\ e_2'^{1-2} &= e_1 & & & & & (i = 4, \dots, n). \end{aligned}$$

By the associative law, $e_1'^2 e_2 = a_{12} e_1' e_3 = a_{12} c_{13} e_n = b_{11} e_{n-1} e_2 = b_{11} e_n$ and hence $b_{11} = a_{12} c_{13}$. Similarly from the products $e_2 e_1 e_2$ and $e_2^2 e_1$ we obtain $b_{21} = a_{12} c_{23}$, $c_{41} = c_{23}(a_{12} + a_{21})$. Hence we write (75) in the form

$$(76) \quad \begin{aligned} e_1'^2 &= a_{11} e_3' + a_{12} c_{13} e_{n-1} + c_{11} e_n, & e_1' e_2 &= a_{12} e_3', \\ e_2' e_1 &= a_{21} e_3' + a_{12} c_{23} e_{n-1} + c_{21} e_n, & e_1' e_3 &= c_{13} e_n, \\ e_3' e_1 &= c_{31} e_n, & e_2 e_3 &= c_{23} e_n, & e_4 e_1 &= c_{23}(a_{12} + a_{21}) e_n, \\ e_3'^2 &= c_{33} e_n, & e_2'^{1-2} &= e_1 & & & (i = 4, \dots, n). \end{aligned}$$

Since $e_1'^2 e_3 = 0 = a_{11} e_3'^2$, $e_1' e_2 e_3 = 0 = a_{12} e_3'^2$, $e_2 e_1 e_3 = 0 = a_{21} e_3'^2$, we have the relations

$$(77) \quad a_{11} e_3'^2 = a_{12} e_3'^2 = a_{21} e_3'^2 = 0.$$

Now suppose that $e_3^2 \neq 0$ in (76). Then by (77) $a_{11} = a_{12} = a_{21} = 0$ and (76) is now

$$(78) \quad \begin{aligned} e_1^2 &= c_{11}e_n, & e_3^2 &= c_{33}e_n, & e_2e_1 &= c_{21}e_n, & e_1e_3 &= c_{13}e_n, \\ e_3e_1 &= c_{31}e_n, & e_2e_3 &= c_{23}e_n, & e_2^{1-2} &= e_1 & (i = 4, \dots, n). \end{aligned}$$

If we now interchange e_2 and e_3 , (78) becomes a regular algebra which, when simplified by the transformations of Chapter III, becomes algebra 4 or algebra 5, if not reducible. Hence we assume that $e_3^2 = 0$; also that a_{11}, a_{12}, a_{21} are not all zero. Algebra (76) with $e_3^2 = 0$ is algebra 6 or algebra 7 in the summary according as it is commutative or non-commutative.

4. Algebras arising from algebras 6 and 7 in $n - 1$ units by the use of Theorem 7₁ with $s = n - 1$. The new algebras have the form

$$(79) \quad \begin{aligned} e_1^2 &= a_{11}e_3 + b_{11}e_{n-2} + c_{11}e_{n-1} + d_{11}e_n, \\ e_1e_2 &= a_{12}e_3 + d_{12}e_n, \\ e_2e_1 &= a_{21}e_3 + b_{21}e_{n-2} + c_{21}e_{n-1} + d_{21}e_n, \\ e_1e_3 &= c_{13}e_{n-1} + d_{13}e_n, & e_3e_1 &= c_{31}e_{n-1} + d_{31}e_n, \\ e_2e_3 &= c_{23}e_{n-1} + d_{23}e_n, & e_3e_2 &= d_{32}e_n, \\ e_1e_4 &= d_{14}e_n, & e_4e_1 &= c_{41}e_{n-1} + d_{41}e_n, \\ e_1e_i &= d_{1i}e_n, & e_ie_1 &= d_{i1}e_n & (i = 5, \dots, n-1), \\ e_3e_i &= d_{3i}e_n, & e_ie_3 &= d_{i3}e_n & (i = 4, \dots, n-1), \\ e_3^2 &= d_{33}e_n, & e_2^{1-2} &= e_1 & (i = 4, \dots, n). \end{aligned}$$

The associative law gives $e_2e_3e_2 = 0 = c_{23}e_{n-1}e_2 = c_{23}e_n$. Hence $c_{23} = 0$. Also $e_3e_2^2 = 0 = e_3e_4$, $e_2^2e_3 = 0 = e_4e_3$ and therefore $e_3e_2^3 = 0 = e_3e_5$, ..., $e_3e_2^{n-3} = 0 = e_3e_{n-1}$, $e_2^3e_3 = 0 = e_3e_5$, ..., $e_2^{n-3}e_3 = 0 = e_{n-1}e_3$. Then $e_4e_1e_2 = 0 = c_{41}e_{n-1}e_2 = c_{41}e_n$, and hence $c_{41} = 0$. Also $e_1e_4e_2 = 0 = e_1e_2^3 = e_1e_5$, $e_2e_4e_1 = 0 =$

$e_2^3 e_1 = e_5 e_1$, $e_1 e_2^4 = 0 = e_1 e_6$, ..., $e_1 e_2^{n-3} = 0 = e_1 e_{n-1}$,
 $e_2^4 e_1 = 0 = e_6 e_1$, ..., $e_2^{n-3} e_1 = e_{n-1} e_1$. From the products
 $e_1 e_3 e_2$, $e_2 e_3 e_1$, $e_4 e_2 e_1$, $e_4 e_1^2$, we find $c_{13} = c_{31} = b_{21} = b_{11} = 0$.
 The algebras (79) are now of the same form as (72) with (73) and
 (74). Hence we again obtain algebras 6 and 7.

5. Algebras derived from algebras 4 and 5 in $n - 1$ units
by Theorem 6₂ with $s = n - 1$, but such that e_1 or e_2 in the new
algebras is of degree $n - 2$. The new algebras, if such exist,
 are of the form

$$\begin{aligned} e_1^2 &= a_{11} e_{n-1} + b_{11} e_n, & e_2^2 &= a_{22} e_{n-1} + b_{22} e_n, \\ e_1 e_2 &= a_{12} e_{n-1} + b_{12} e_n, & e_2 e_1 &= a_{21} e_{n-1} + b_{21} e_n, \\ e_2 e_3 &= & b_{23} e_n, & e_3 e_2 &= a_{32} e_{n-1} + b_{32} e_n, \\ e_1 e_1 &= b_{11} e_n, & e_1 e_1 &= b_{11} e_n & (i = 3, \dots, n-1), \\ e_2 e_1 &= b_{21} e_n, & e_1 e_2 &= b_{12} e_n & (i = 4, \dots, n-1), \\ e_3^{1-2} &= e_1 & & & (i = 3, \dots, n-1) \end{aligned}$$

where e_1 or e_2 is of degree $n - 2$. Since $e_1 e_3^2 = 0 = e_1 e_4$, we
 have $e_1 e_3^{n-3} = 0 = e_1 e_{n-1}$, and similarly, $e_2 e_3^{n-3} = 0 = e_2 e_{n-1}$.
 Hence $e_1^3 = 0 = e_2^3$, that is, e_1 and e_2 are each of degree ≤ 2 .
 But $n - 2 \leq 2$ for $n \leq 4$, and the case considered in this section
 cannot arise unless n is greater than 4.

6. Algebras arising from algebras 6 and 7 in $n - 1$ units
by the use of Theorem 7₂ with $s = n - 1$, but such that e_1 in the
new algebras is of degree $n - 2$. For $n > 5$ these algebras have
 the form (79) except that the powers of e_2 are now $e_2^{1-2} = e_1$
 $(i = 4, \dots, n-1)$, $e_2^{n-2} = 0$. The proof in Section 4 that
 $e_1 e_{n-1} = 0$ holds in this case also. Hence $e_1^3 = a_{11} e_1 e_3 + b_{11} e_1 e_{n-2}$
 $= a_{11} c_{13} e_{n-1} + (a_{11} d_{13} + b_{11} d_{1, n-2}) e_n$ and $e_1^4 = 0$, e_1 is of degree
 3 at most. But $n - 2 \leq 3$ for $n \leq 5$. The case considered in this

section cannot arise for $n < 5$, hence the case $n = 5$ is the only one to be considered. By Theorem 7₂, the new algebras have the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5, & e_1^3 &\neq 0, \\
 e_1e_2 &= a_{12}e_3 + c_{12}e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5, \\
 (80) \quad e_1e_3 &= c_{13}e_5, & e_3e_1 &= c_{31}e_5, & e_1e_4 &= c_{14}e_5, & e_4e_1 &= c_{41}e_5, \\
 e_2e_3 &= c_{23}e_5, & e_3e_2 &= c_{32}e_5, & e_3e_4 &= c_{34}e_5, & e_4e_3 &= c_{43}e_5, \\
 e_3^2 &= c_{33}e_5, & e_2^2 &= e_4.
 \end{aligned}$$

By the associative law $e_2^2e_3 = 0 = e_4e_3$, $e_3e_2^2 = 0 = e_3e_4$. Also, just as in Section 3, we have (77). The products $e_1e_2^2$ and $e_2^2e_1$ give

$$(81) \quad c_{14} = a_{12}c_{32}, \quad c_{41} = a_{21}c_{23}.$$

If $e_3^2 \neq 0$, then by (77) we have $a_{11} = a_{12} = a_{21} = 0$, and in view of (81), $e_1e_4 = 0 = e_4e_1$. Hence $e_1^3 = 0$, contrary to the hypothesis that e_1 is of degree $n - 2$. Therefore $e_3^2 = 0$. From

$$\begin{aligned}
 (82) \quad e_1^3 &= a_{11}e_1e_3 + b_{11}e_1e_4 = (a_{11}c_{13} + b_{11}a_{12}c_{32})e_5 \\
 &= a_{11}e_3e_1 + b_{11}e_4e_1 = (a_{11}c_{31} + b_{11}a_{21}c_{23})e_5 \neq 0
 \end{aligned}$$

we have

$$(83) \quad a_{11}(c_{13} - c_{31}) + b_{11}(a_{12}c_{32} - a_{21}c_{23}) = 0.$$

If $a_{11} = b_{11} = 0$, then $e_1^3 = 0$. Hence a_{11} and b_{11} are not both zero.

First let $b_{11} = 0$. Then $a_{11} \neq 0$ and from (83) and (82) we have $c_{13} = c_{31} \neq 0$. Put $e_1' = -a_{11}^{-1}a_{21}e_1 + e_2$, $e_2' = e_1$, $e_3' = e_4$, $e_4' = a_{11}e_3 + c_{11}e_5$, $e_5' = a_{11}c_{13}e_5$ and obtain $e_2e_4 = e_4e_2 = e_5$, algebras (80) now taking the form of (72) with $n = 5$ and $e_3e_4 = 0 = e_4e_3 = e_3^2$. Hence we obtain algebras 6 and 7 in 5 units.

Now suppose $b_{11} \neq 0$. Put $e_1' = -b_{11}^{-1}b_{21}e_1 + e_2$, $e_2' = e_1$, $e_4' = a_{11}e_3 + b_{11}e_4 + c_{11}e_5$, $e_5' = (a_{11}c_{13} + b_{11}a_{12}c_{32})e_5$. Again algebras (80) take the form of (72) with $n = 5$ and $e_3e_4 = 0 = e_4e_3 = e_3^2$, and we ultimately have algebras 6 and 7.

Observing that algebras (70) and (71) are included as special cases in algebras 6 and 7 respectively, we may summarize our results in the following

THEOREM. All the algebras of order $n > 4$ of degree $n - 2$ are included in the classes of algebras 4, 5, 6 and 7 and their reciprocals.

7. Relations between the coefficients in a certain irregular algebra. We observe that in a regular algebra the associative law gives us no relations whatever between the coefficients. But in the case of some of the irregular algebras there are numerous relations other than those which appear in the final classification. We shall not attempt to compute all of these relations, but by way of illustration, shall consider some of those existing among the coefficients of algebra 7.

First we consider the general algebra 7 with $n > 5$. In this case $e_1^3 = ae_1e_3 = ae_3e_1$, that is, $age_n = ahe_n$. Hence if $a \neq 0$ then $g = h$. From the product $e_1e_2e_1$ we find

$$(84) \quad ch = dg.$$

Therefore if $a \neq 0$, $g \neq 0$, then $g = h$, $c = d$. Now $e_2e_1^2 = ae_2e_3 + ce_2e_{n-1} = (ak + cg)e_n = de_3e_1 = dhe_n$ and

$$(85) \quad ak + cg = dh.$$

Hence for $a \neq 0$, we have $k = 0$. Thus in any case $ak = 0$ for $n > 5$. Then if c, d, g, h are all different from zero, from

(84) and (85) we have $h/g = g/h$, $g^2 = h^2$, $g = \pm h$ and $c = \pm d$.

Now let $n = 5$. We have $e_1^3 = ae_1e_3 = age_5 = ae_3e_1 + cge_4e_1 = [ah + cgk(c + d)]e_5$ and therefore

$$(86) \quad ag = ah + cgk(c + d).$$

From the product $e_2e_1^2$ we have

$$(87) \quad ak + cg = dh + ck^2(c + d).$$

Since arbitrarily chosen elements in an arbitrary field may not satisfy (86) and (87) these relations may be a restriction on the generality of algebra 7 in 5 units.

CHAPTER V

ALGEBRAS OF DEGREE $n - 3$ ($n > 4$)

1. Regular algebras of degree $n - 3$. We shall call the classes of algebras of order 5 derived from the zero algebra in 4 units by the use of Theorem 5 algebras 8 and 9. The discussion of regular algebras in Chapter III shows that these two algebras have the forms given in the classification of algebras in 5 units; also that we finally have, by induction, the regular algebras 8 and 9 in n units.

2. Algebras arising from algebras 4 and 5 in $n - 1$ units by the use of Theorem 6₂. The new algebras take the form

$$\begin{aligned}
 (88) \quad & e_1^2 = a_{11}e_{n-1} + b_{11}e_n, & e_2^2 &= a_{22}e_{n-1} + b_{22}e_n, \\
 & e_1e_2 = a_{12}e_{n-1} + b_{12}e_n, & e_2e_1 &= a_{21}e_{n-1} + b_{21}e_n, \\
 & e_2e_3 = & b_{23}e_n, & e_3e_2 &= a_{32}e_{n-1} + b_{32}e_n, \\
 & e_1e_i = b_{1i}e_n, & e_ie_1 &= b_{i1}e_n & (i = 3, \dots, n-1), \\
 & e_2e_i = b_{2i}e_n, & e_ie_2 &= b_{i2}e_n & (i = 4, \dots, n-1), \\
 & e_3^{i-2} = e_1 & & & (i = 3, \dots, n-1).
 \end{aligned}$$

By the associative law $e_1e_3^2 = 0 = e_1e_4$, $e_1e_3^3 = 0 = e_1e_5$, ..., $e_1e_3^{n-3} = 0 = e_1e_{n-1}$, $e_3^2e_1 = 0 = e_4e_1$, ..., $e_3^{n-3}e_1 = 0 = e_{n-1}e_1$, and similarly, $e_2e_1 = 0 = e_4e_2$ ($i = 4, \dots, n-1$). If $b_{23} \neq 0$, put $e_1' = e_1 - b_{23}^{-1}b_{13}e_2$ and obtain $e_1'e_3 = 0$, $e_3e_1 = a_{31}e_{n-1} + b_{31}e_n$. The algebras may now be written in the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_{n-1} + b_{11}e_n, & e_2^2 &= a_{22}e_{n-1} + b_{22}e_n, \\
 e_1e_2 &= a_{12}e_{n-1} + b_{12}e_n, & e_2e_1 &= a_{21}e_{n-1} + b_{21}e_n, \\
 (89) \quad e_3e_1 &= a_{31}e_{n-1} + b_{31}e_n, & e_2e_3 &= b_{23}e_n, \\
 e_3e_2 &= a_{32}e_{n-1} + b_{32}e_n, \\
 e_3^{i-2} &= e_1 & (i = 3, \dots, n-1).
 \end{aligned}$$

If in (88) $b_{23} = 0$, we interchange e_1 and e_2 and have (89) with $a_{32} = 0$.

For algebras (89) the linear sets B_i are $B_1 = (e_1, e_2, e_3)$, $B_2 = (e_4, e_n)$, $B_3 = (e_5)$, $B_4 = (e_6)$, ..., $B_{n-3} = (e_{n-1})$. We subject the basal units to the cyclic permutation $(e_4e_5 \cdots e_{n-1}e_n)$ and have, in the commutative case,

$$\begin{aligned}
 (90) \quad e_1^2 &= ae_4 + be_n, & e_2^2 &= he_4 + je_n, & e_1e_2 &= ce_4 + de_n = e_2e_1, \\
 e_2e_3 &= me_4 = e_3e_2, & e_3^{i-3} &= e_1 & (i = 5, \dots, n)
 \end{aligned}$$

and in the non-commutative case,

$$\begin{aligned}
 (91) \quad e_1^2 &= ae_4 + be_n, & e_2^2 &= he_4 + je_n, & e_1e_2 &= ce_4 + de_n, \\
 e_2e_1 &= fe_4 + ge_n, & e_3e_1 &= ke_4 + le_n, & e_2e_3 &= me_4, \\
 e_3e_2 &= pe_4 + qe_n, & e_3^{i-3} &= e_1 & (i = 5, \dots, n).
 \end{aligned}$$

The sets B_i are now $B_1 = (e_1, e_2, e_3)$, $B_2 = (e_4, e_5)$, $B_3 = (e_6)$, $B_4 = (e_7)$, ..., $B_{n-3} = (e_n)$.

3. Algebras derived from (90) and (91) by means of Theorem 7₁ with $s = n - 1$. The new algebras take the form

$$\begin{aligned}
e_1^2 &= a_{11}e_4 + b_{11}e_{n-1} + c_{11}e_n, & e_2^2 &= a_{22}e_4 + b_{22}e_{n-1} + c_{22}e_n, \\
e_1e_2 &= a_{12}e_4 + b_{12}e_{n-1} + c_{12}e_n, & e_2e_1 &= a_{21}e_4 + b_{21}e_{n-1} + c_{21}e_n, \\
e_1e_3 &= & c_{13}e_n, & e_3e_1 &= a_{31}e_4 + b_{31}e_{n-1} + c_{31}e_n, \\
(92) \quad e_2e_3 &= a_{23}e_4 & + c_{23}e_n, & e_3e_2 &= a_{32}e_4 + b_{32}e_{n-1} + c_{32}e_n, \\
e_1e_j &= c_{1j}e_n, & e_je_1 &= c_{j1}e_n & (i = 1, 2; j = 4, \dots, n-1), \\
e_4e_1 &= c_{41}e_n, & e_1e_4 &= c_{14}e_n & (i = 3, \dots, n-1), \\
e_3^{i-3} &= e_1 & & & (i = 5, \dots, n).
\end{aligned}$$

By the associative law $e_4e_3^2 = 0 = e_4e_5$, ..., $e_4e_3^{n-4} = 0 = e_4e_{n-1}$,
 $e_3^2e_4 = 0 = e_5e_4$, ..., $e_3^{n-4}e_4 = 0 = e_{n-1}e_4$, $e_1e_3^2 = 0 = e_1e_5$,
 $e_1e_3^3 = 0 = e_1e_6$, ..., $e_1e_3^{n-4} = 0 = e_1e_{n-1}$, $e_3^3e_1 = e_3e_5e_1 = 0$
 $= e_6e_1$, ..., $e_3^{n-4}e_1 = 0 = e_{n-1}e_1$, $e_2e_3^3 = e_2e_5e_3 = 0 = e_2e_6$, ...,
 $e_2e_3^{n-4} = 0 = e_2e_{n-1}$, $e_3^3e_2 = e_3e_5e_2 = 0 = e_6e_2$, ..., $e_3^{n-4}e_2 = 0$
 $= e_{n-1}e_2$ and we have

$$(93) \quad e_4e_1 = 0 = e_1e_4 \quad (i = 5, \dots, n-1),$$

$$(94) \quad e_1e_5 = 0$$

$$(95) \quad e_1e_j = 0 = e_je_1 \quad (i = 1, 2; j = 6, \dots, n-1).$$

We note that in (95) j has only the value 6 for $n = 6$. Now if
 $n > 6$ apply to the algebras the transformation

$$\begin{aligned}
(96) \quad e_1' &= e_1 - c_{13}e_{n-1}, & e_2' &= e_2 - a_{23}c_{43}e_{n-2} - c_{23}e_{n-1}, \\
& & e_4' &= e_4 - c_{43}e_{n-1}.
\end{aligned}$$

For $n = 6$, replace the second equation of (96) by $e_2' = e_2 -$
 $a_{23}c_{43}e_3 - c_{23}e_5$. In either case we have $e_1e_3 = 0$, $e_2e_3 = a_{23}e_4$,
 $e_4e_3 = 0$ and we now write the algebras in the form

$$\begin{aligned}
e_1^2 &= a_{11}e_4 + b_{11}e_{n-1} + c_{11}e_n, & e_2^2 &= a_{22}e_4 + b_{22}e_{n-1} + c_{22}e_n, \\
e_1e_2 &= a_{12}e_4 + b_{12}e_{n-1} + c_{12}e_n, & e_2e_1 &= a_{21}e_4 + b_{21}e_{n-1} + c_{21}e_n, \\
e_3e_1 &= a_{31}e_4 + b_{31}e_{n-1} + c_{31}e_n, & e_2e_3 &= a_{23}e_4, \\
(97) \quad e_3e_2 &= a_{32}e_4 + b_{32}e_{n-1} + c_{32}e_n, & e_1e_4 &= c_{14}e_n, & e_4e_1 &= c_{41}e_n, \\
e_2e_4 &= c_{24}e_n, & e_4e_2 &= c_{42}e_n, & e_3e_4 &= c_{34}e_n, & e_4^2 &= c_{44}e_n, \\
e_5e_1 &= c_{51}e_n, & e_2e_5 &= c_{25}e_n, & e_5e_2 &= c_{52}e_n, \\
e_3^{1-3} &= e_1 & (i = 5, \dots, n).
\end{aligned}$$

By the associative law $e_2e_3 = 0 = e_2e_5$; $e_1^2e_3 = 0 = b_{11}e_{n-1}e_3 = b_{11}e_n$ and $b_{11} = 0$; from $e_2e_1e_3$ we have $b_{21} = 0$, and from $e_3e_1e_3$, $b_{31} = 0$. From the products $e_3^2e_1$, $e_1e_2e_3$, $e_2^2e_3$, $e_3e_2e_3$, $e_3^2e_2$ we have $c_{51} = a_{31}c_{34}$, $b_{12} = a_{23}c_{14}$, $b_{22} = a_{23}c_{24}$, $b_{32} = a_{23}c_{34}$, $c_{52} = c_{34}(a_{23} + a_{32})$. The algebras (97) may now be written in the form

$$\begin{aligned}
e_1^2 &= a_{11}e_4 + c_{11}e_n, & e_2^2 &= a_{22}e_4 + a_{23}c_{24}e_{n-1} + c_{22}e_n, \\
e_1e_2 &= a_{12}e_4 + a_{23}c_{14}e_{n-1} + c_{12}e_n, & e_2e_1 &= a_{21}e_4 + c_{21}e_n, \\
e_3e_1 &= a_{31}e_4 + c_{31}e_n, & e_2e_3 &= a_{23}e_4, \\
(98) \quad e_3e_2 &= a_{32}e_4 + a_{23}c_{34}e_{n-1} + c_{32}e_n, & e_1e_4 &= c_{14}e_n, \\
e_4e_1 &= c_{41}e_n, & e_2e_4 &= c_{24}e_n, & e_4e_2 &= c_{42}e_n, & e_3e_4 &= c_{34}e_n, \\
e_4^2 &= c_{44}e_n, & e_5e_1 &= a_{31}c_{34}e_n, & e_5e_2 &= c_{34}(a_{23} + a_{32})e_n, \\
e_3^{1-3} &= e_1 & (i = 5, \dots, n).
\end{aligned}$$

Again, using the associative law, we find $e_1^2e_4 = 0 = a_{11}e_4^2$, $e_1e_2e_4 = 0 = a_{12}e_4^2$, $e_2e_1e_4 = 0 = a_{21}e_4^2$, $e_2^2e_4 = 0 = a_{22}e_4^2$, $e_2e_3e_4 = 0 = a_{23}e_4^2$, $e_3e_1e_4 = 0 = a_{31}e_4^2$, $e_3e_2e_4 = 0 = a_{32}e_4^2$, and hence

$$\begin{aligned}
(99) \quad a_{11}e_4^2 &= a_{22}e_4^2 = a_{12}e_4^2 = a_{21}e_4^2 \\
&= a_{31}e_4^2 = a_{23}e_4^2 = a_{32}e_4^2 = 0.
\end{aligned}$$

Now suppose $e_4^2 \neq 0$. Then by (99) we have $a_{11} = a_{22} = a_{12} = a_{21} = a_{31} = a_{23} = a_{32} = 0$ and the algebras (98) become

$$\begin{aligned}
 (100) \quad & e_1^2 = c_{11}e_n, \quad e_2^2 = c_{22}e_n, \quad e_1e_2 = c_{12}e_n, \quad e_2e_1 = c_{21}e_n, \\
 & e_3e_1 = c_{31}e_n, \quad e_3e_2 = c_{32}e_n, \quad e_1e_4 = c_{14}e_n, \quad e_4e_1 = c_{41}e_n, \\
 & e_2e_4 = c_{24}e_n, \quad e_4e_2 = c_{42}e_n, \quad e_3e_4 = c_{34}e_n, \quad e_4^2 = c_{44}e_n, \\
 & e_3^{1-3} = e_1 \quad (i = 5, \dots, n).
 \end{aligned}$$

If we interchange e_3 and e_4 in (100) we obtain a regular algebra which, by means of the transformations of Chapter III, takes the form of algebra 8 or algebra 9 in n units.

Next let $e_4^2 = 0$. If (98) is commutative, we have algebra 10 in the final classification of algebras in n units. In the non-commutative case we have algebra 11 in n units. In view of the origin of these algebras, n must be greater than 5.

4. Algebras arising from algebras 10 and 11 in $n - 1$ units by the use of Theorem 7₁ with $s = n - 1$. The new algebras take the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_4 + c_{11}e_{n-1} + d_{11}e_n, \\
 e_2^2 &= a_{22}e_4 + b_{22}e_{n-2} + c_{22}e_{n-1} + d_{22}e_n, \\
 e_1e_2 &= a_{12}e_4 + b_{12}e_{n-2} + c_{12}e_{n-1} + d_{12}e_n, \\
 e_2e_1 &= a_{21}e_4 + c_{21}e_{n-1} + d_{21}e_n, \quad e_1e_3 = d_{13}e_n, \\
 e_3e_1 &= a_{31}e_4 + c_{31}e_{n-1} + d_{31}e_n, \quad e_2e_3 = a_{23}e_4 + d_{23}e_n, \\
 e_3e_2 &= a_{32}e_4 + b_{32}e_{n-2} + c_{32}e_{n-1} + d_{32}e_n, \\
 e_1e_4 &= c_{14}e_{n-1} + d_{14}e_n, \quad e_4e_1 = c_{41}e_{n-1} + d_{41}e_n, \\
 e_2e_4 &= c_{24}e_{n-1} + d_{24}e_n, \quad e_4e_2 = c_{42}e_{n-1} + d_{42}e_n, \\
 e_3e_4 &= c_{34}e_{n-1} + d_{34}e_n, \quad e_4e_3 = d_{43}e_n, \quad e_1e_5 = d_{15}e_n, \\
 e_5e_1 &= c_{51}e_{n-1} + d_{51}e_n, \quad e_2e_5 = d_{25}e_n, \quad e_5e_2 = c_{52}e_{n-1} + d_{52}e_n, \\
 e_1e_j &= d_{1j}e_n, \quad e_je_1 = d_{j1}e_n \quad (i = 1, 2; j = 6, \dots, n-1), \\
 e_4e_1 &= d_{41}e_n, \quad e_1e_4 = d_{14}e_n \quad (i = 4, \dots, n-1), \\
 e_3^{1-3} &= e_1 \quad (i = 5, \dots, n).
 \end{aligned}$$

It is evident that we now have $n > 6$. Using the associative law we have $e_3e_4e_3 = 0 = c_{34}e_{n-1}e_3 = c_{34}e_n$, $c_{34} = 0$. Then just as in Section 3, we have (93). Also from the products $e_5e_1e_3$ and $e_5e_2e_3$ we have $c_{51} = 0 = c_{52}$. Then (94) and (95) are found to hold. Using the products $e_1e_4e_3$, $e_4e_1e_3$, $e_2e_4e_3$, $e_3e_4e_2$, $e_1e_2e_5$, $e_2^2e_5$, $e_3e_2e_5$ we find $c_{14} = c_{41} = c_{24} = c_{42} = b_{12} = b_{22} = b_{32} = 0$. The algebras are now in the form (92) with (93), (94), (95). Hence we again obtain algebras 10 and 11 in n units.

5. Algebras arising from algebras 6 and 7 in $n - 1$ units by the use of Theorem 7₂ with $s = n - 1$ ($n > 5$). The new algebras are

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + a_{12}c_{13}e_{n-2} + c_{11}e_{n-1} + d_{11}e_n, \\
 e_1e_2 &= a_{12}e_3 + d_{12}e_n, \\
 e_2e_1 &= a_{21}e_3 + a_{12}c_{23}e_{n-2} + c_{21}e_{n-1} + d_{21}e_n, \\
 e_1e_3 &= c_{13}e_{n-1} + d_{13}e_n, & e_3e_1 &= c_{31}e_{n-1} + d_{31}e_n, \\
 (101) \quad e_2e_3 &= c_{23}e_{n-1} + d_{23}e_n, & e_3e_2 &= d_{32}e_n, \\
 e_1e_4 &= d_{14}e_n, & e_4e_1 &= c_{23}(a_{12} + a_{21})e_{n-1} + d_{41}e_n, \\
 e_1e_i &= d_{1i}e_n, & e_ie_1 &= d_{i1}e_n \quad (i = 5, \dots, n-1), \\
 e_3e_i &= d_{3i}e_n, & e_ie_3 &= d_{i3}e_n \quad (i = 3, \dots, n-1), \\
 e_2^{i-2} &= e_1 & & (i = 4, \dots, n-1).
 \end{aligned}$$

Using the associative law we find

$$\begin{aligned}
 (102) \quad e_3e_1 &= 0 = e_1e_3 \quad (i = 4, \dots, n-1), \\
 e_1e_i &= 0 = e_ie_1 \quad (i = 5, \dots, n-1).
 \end{aligned}$$

Also from the products $e_1e_2^2$ and $e_2^2e_1$ we have

$$(103) \quad d_{14} = a_{12}d_{32}, \quad d_{41} = a_{21}d_{23}.$$

Just as in Chapter IV, Section 3, we find

$$(77) \quad a_{11}e_3^2 = a_{12}e_3^2 = a_{21}e_3^2 = 0.$$

We first consider the case $e_3^2 \neq 0$. By (77), $a_{11} = a_{12} = a_{21} = 0$ and the algebras assume the form

$$\begin{aligned}
 e_1^2 &= c_{11}e_{n-1} + d_{11}e_n, \\
 e_1e_2 &= d_{12}e_n, & e_2e_1 &= c_{21}e_{n-1} + d_{21}e_n, \\
 (104) \quad e_1e_3 &= c_{13}e_{n-1} + d_{13}e_n, & e_3e_1 &= c_{31}e_{n-1} + d_{31}e_n, \\
 e_2e_3 &= c_{23}e_{n-1} + d_{23}e_n, & e_3e_2 &= d_{32}e_n, & e_3^2 &= d_{33}e_n, \\
 e_2^{i-2} &= e_1 & (i &= 4, \dots, n-1).
 \end{aligned}$$

If in (104) $d_{32} \neq 0$, put $e_1' = e_1 - d_{32}^{-1}d_{12}e_3$ and obtain

$$(105) \quad e_1e_2 = 0.$$

Now make the cyclic permutations (e_2e_3) and $(e_4e_5 \dots e_n)$ of the basal units. The algebras (104) with (105) take the form of (90) or (91). In case $d_{32} = 0$ in (104), we subject the basal units to the cyclic permutations $(e_1e_2e_3)$ and $(e_4e_5 \dots e_n)$, and again obtain (90) or (91).

We now assume $e_3^2 = 0$ and that a_{11}, a_{12}, a_{21} are not all zero in the algebras (101) with (102) and (103). The linear sets B_i are $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_4)$, $B_3 = (e_5, e_n)$, $B_4 = (e_6)$, $B_5 = (e_7)$, ..., $B_{n-3} = (e_{n-1})$. If $n = 6$, we interchange e_5 and e_6 and write the algebras in the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + a_{12}c_{13}e_4 + d_{11}e_5 + c_{11}e_6, \\
 e_1e_2 &= a_{12}e_3 + d_{12}e_5, \\
 e_2e_1 &= a_{21}e_3 + a_{12}c_{23}e_4 + d_{21}e_5 + c_{21}e_6, \\
 (106) \quad e_1e_3 &= d_{13}e_5 + c_{13}e_6, & e_3e_1 &= d_{31}e_5 + c_{31}e_6, \\
 e_2e_3 &= d_{23}e_5 + c_{23}e_6, & e_3e_2 &= d_{32}e_5, \\
 e_1e_4 &= a_{12}d_{32}e_5, & e_4e_1 &= a_{21}d_{23}e_5 + c_{23}(a_{12} + a_{21})e_6, \\
 e_2^2 &= e_4, & e_2^3 &= e_6
 \end{aligned}$$

whose linear sets B_i are $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_4)$, $B_3 = (e_5, e_6)$. If (106) is commutative we write it in the form

$$(107) \quad \begin{aligned} e_1^2 &= ae_3 + dle_4 + be_5 + ce_6, & e_1e_2 &= de_3 + fe_5 = e_2e_1, \\ e_1e_3 &= ke_5 + le_6 = e_3e_1, & e_2e_3 &= pe_5 = e_3e_2, \\ e_1e_4 &= dpe_5 = e_4e_1, & e_2^2 &= e_4, & e_2^3 &= e_6, \end{aligned}$$

where a and d are not both zero. If (106) is non-commutative we have

$$(108) \quad \begin{aligned} e_1^2 &= ae_3 + dle_4 + be_5 + ce_6, \\ e_1e_2 &= de_3 + fe_5, & e_2e_1 &= ge_3 + dqe_4 + he_5 + je_6, \\ e_1e_3 &= ke_5 + le_6, & e_3e_1 &= me_5 + ve_6, \\ e_2e_3 &= pe_5 + qe_6, & e_3e_2 &= re_5, & e_1e_4 &= dre_5, \\ e_4e_1 &= gpe_5 + q(d+g)e_6, & e_2^2 &= e_4, & e_2^3 &= e_6, \end{aligned}$$

where a, d, g are not all zero. If $n > 6$, the cyclic permutation $(e_5e_6 \cdots e_{n-1}e_n)$ of the basal units gives our algebras (101), with (102), (103), $e_3^2 = 0$, the form

$$(109) \quad \begin{aligned} e_1^2 &= a_{11}e_3 + d_{11}e_5 + a_{12}c_{13}e_{n-1} + c_{11}e_n, \\ e_1e_2 &= a_{12}e_3 + d_{12}e_5, \\ e_2e_1 &= a_{21}e_3 + d_{21}e_5 + a_{12}c_{23}e_{n-1} + c_{21}e_n, \\ e_1e_3 &= d_{13}e_5 + c_{13}e_n, & e_3e_1 &= d_{31}e_5 + c_{31}e_n, \\ e_2e_3 &= d_{23}e_5 + c_{23}e_n, & e_3e_2 &= d_{32}e_5, \\ e_1e_4 &= a_{12}d_{32}e_5, & e_4e_1 &= a_{21}d_{23}e_5 + c_{23}(a_{12} + a_{21})e_n, \\ e_2^2 &= e_4, & e_2^{1-3} &= e_1 \end{aligned} \quad (1 = 6, \dots, n)$$

whose linear sets B_i are now $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_4)$, $B_3 = (e_5, e_6)$, $B_4 = (e_7)$, $B_5 = (e_8)$, $\dots$, $B_{n-3} = (e_n)$. If (109) is commutative we write it in the form

$$\begin{aligned}
 e_1^2 &= ae_3 + be_5 + d\ell e_{n-1} + ce_n, & e_1e_2 &= de_3 + fe_5 = e_2e_1, \\
 (110) \quad e_1e_3 &= ke_5 + \ell e_n = e_3e_1, & e_2e_3 &= pe_5 = e_3e_2, \\
 e_1e_4 &= dpe_5 = e_4e_1, & e_2^2 &= e_4, \quad e_2^{1-3} = e_1 \quad (i = 6, \dots, n).
 \end{aligned}$$

where a and d are not both zero. In the non-commutative case we have

$$\begin{aligned}
 e_1^2 &= ae_3 + be_5 + d\ell e_{n-1} + ce_n, \\
 e_1e_2 &= de_3 + fe_5, & e_2e_1 &= ge_3 + he_5 + dqe_{n-1} + je_n, \\
 (111) \quad e_1e_3 &= ke_5 + \ell e_n, & e_3e_1 &= me_5 + ve_n, & e_2e_3 &= pe_5 + qe_n, \\
 e_3e_2 &= re_5, & e_1e_4 &= dre_5, & e_4e_1 &= gpe_5 + q(d + g)e_n, \\
 e_2^2 &= e_4, & e_2^{1-3} &= e_1 & & (i = 6, \dots, n).
 \end{aligned}$$

with a, d, g not all zero.

6. Algebras arising from algebras (110) and (111) in $n - 1$ units by the use of Theorem 8₁ ($n > 7$). The new algebras have the multiplication table

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_5 + c_{11}e_{n-2} + d_{11}e_{n-1} + g_{11}e_n, \\
 e_1e_2 &= a_{12}e_3 + b_{12}e_5 + g_{12}e_n, \\
 e_2e_1 &= a_{21}e_3 + b_{21}e_5 + c_{21}e_{n-2} + d_{21}e_{n-1} + g_{21}e_n, \\
 e_1e_3 &= b_{13}e_5 + d_{13}e_{n-1} + g_{13}e_n, \\
 e_3e_1 &= b_{31}e_5 + d_{31}e_{n-1} + g_{31}e_n, \\
 (112) \quad e_2e_3 &= b_{23}e_5 + d_{23}e_{n-1} + g_{23}e_n, & e_3e_2 &= b_{32}e_5 + g_{32}e_n, \\
 e_1e_4 &= b_{14}e_5 + g_{14}e_n, & e_4e_1 &= b_{41}e_5 + d_{41}e_{n-1} + g_{41}e_n, \\
 e_1e_1 &= g_{11}e_n, & e_1e_1 &= g_{11}e_n & (i = 5, \dots, n - 1), \\
 e_3e_1 &= g_{31}e_n, & e_1e_3 &= g_{13}e_n & (i = 3, \dots, n - 1), \\
 e_5e_1 &= g_{51}e_n, & e_1e_5 &= g_{15}e_n & (i = 2, 4, 5, \dots, n - 1), \\
 e_2^2 &= e_4 & e_2^{1-3} &= e_1 & (i = 6, \dots, n).
 \end{aligned}$$

Using the associative law we find

$$\begin{aligned}
 (113) \quad & e_1 e_1 = 0 = e_1 e_1 & (i = 7, \dots, n-1), \\
 & e_3 e_1 = 0 = e_1 e_3 & (i = 6, \dots, n-1), \\
 & e_5 e_1 = 0 = e_1 e_5 & (i = 4, 6, \dots, n-1).
 \end{aligned}$$

If $n > 8$, apply to the algebras the transformation

$$\begin{aligned}
 (114) \quad & e_1' = e_1 - a_{12}b_{32}g_{52}e_{n-3} - (a_{12}g_{32} + b_{12}g_{52})e_{n-2} - g_{12}e_{n-1}, \\
 & e_3' = e_3 - b_{32}g_{52}e_{n-2} - g_{32}e_{n-1}, \quad e_5' = e_5 - g_{52}e_{n-1}.
 \end{aligned}$$

If $n = 8$, replace e_{n-3} by e_4 in (114). In either case the effect of the transformation is the elimination of the coefficients of e_n in the products $e_1 e_2$, $e_3 e_2$, $e_5 e_2$, the apparent introduction of a term in e_{n-1} in $e_1 e_4$, together with other results not evident until we apply the associative law. Using this law we find, from $e_3 e_2^2$ and $e_1 e_2 e_4$, that $e_3 e_4 = 0 = e_1 e_6$. From $e_1 e_2^2$ we have $b_{14} = a_{12}b_{32}$, $d_{14} = 0 = g_{14}$. The products $e_1 e_3 e_2$, $e_3 e_1 e_2$, $e_2 e_3 e_2$ give $d_{13} = b_{32}g_{15}$, $d_{31} = a_{12}g_{33} + b_{12}g_{35}$, $d_{23} = b_{32}g_{25}$. From $e_1^2 e_2$ we have $d_{11} = a_{12}g_{13} + b_{12}g_{15}$ and $c_{11} = a_{12}b_{32}g_{15}$. From $e_2 e_1 e_2$, $d_{21} = a_{12}g_{23} + b_{12}g_{25}$ and $c_{21} = a_{12}b_{32}g_{25}$. The products $e_2^2 e_3$, $e_4 e_2 e_1$, $e_2^2 e_1$ give $g_{43} = g_{25}(b_{23} + b_{32})$, $g_{61} = g_{25}(a_{21}b_{23} + a_{21}b_{32} + a_{12}b_{32})$, $b_{41} = a_{21}b_{23}$, $d_{41} = b_{32}g_{25}(a_{12} + a_{21})$, $g_{41} = g_{23}(a_{12} + a_{21}) + g_{25}(b_{12} + b_{21})$. The algebras may now be written in the form

$$\begin{aligned}
 (115) \quad & e_1'^2 = a_{11}e_3 + b_{11}e_5 + a_{12}b_{32}g_{15}e_{n-2} + (a_{12}g_{13} + b_{12}g_{15})e_{n-1} \\
 & \quad + g_{11}e_n, \quad e_1 e_2 = a_{12}e_3 + b_{12}e_5, \\
 & e_2 e_1 = a_{21}e_3 + b_{21}e_5 + a_{12}b_{32}g_{25}e_{n-2} + (a_{12}g_{23} + b_{12}g_{25})e_{n-1} \\
 & \quad + g_{21}e_n, \quad e_1 e_3 = b_{13}e_5 + b_{32}g_{15}e_{n-1} + g_{13}e_n, \\
 & e_3 e_1 = b_{31}e_5 + (a_{12}g_{33} + b_{12}g_{35})e_{n-1} + g_{31}e_n, \\
 & e_1 e_4 = a_{12}b_{32}e_5, \quad e_4 e_1 = a_{21}b_{23}e_5 + b_{32}g_{25}(a_{12} + a_{21})e_{n-1} \\
 & \quad + [g_{23}(a_{12} + a_{21}) + g_{25}(b_{12} + b_{21})]e_n,
 \end{aligned}$$

$$\begin{aligned}
 e_2 e_3 &= b_{23} e_5 + b_{32} g_{25} e_{n-1} + g_{23} e_n, & e_3 e_2 &= b_{32} e_5, \\
 e_1 e_5 &= g_{15} e_n, & e_5 e_1 &= g_{51} e_n, & e_2 e_5 &= g_{25} e_n, & e_3 e_5 &= g_{35} e_n, \\
 (115) \quad e_5 e_3 &= g_{53} e_n, & e_4 e_3 &= g_{25} (b_{23} + b_{32}) e_n, \\
 e_6 e_1 &= g_{25} (a_{21} b_{23} + a_{21} b_{32} + a_{12} b_{32}) e_n, & e_5^2 &= g_{55} e_n, \\
 e_3^2 &= g_{33} e_n, & e_2^2 &= e_4, & e_2^{1-3} &= e_1 & (i = 6, \dots, n).
 \end{aligned}$$

Note that the product $e_2 e_3 e_1$ gives a second expression for d_{31} ,

$$(116) \quad d_{31} = b_{23} g_{51} - b_{31} g_{25}.$$

Applying the associative law to the products $e_5 e_1 e_3$, $e_5 e_3 e_1$,

$e_5 e_2 e_3$, $e_5 e_3 e_2$ we have

$$(117) \quad b_{13} e_5^2 = b_{31} e_5^2 = b_{23} e_5^2 = b_{32} e_5^2 = 0.$$

From $e_5 e_1^2$, $e_5 e_1 e_2$, $e_5 e_2 e_1$ we obtain

$$(118) \quad a_{11} g_{53} = -b_{11} g_{55}, \quad a_{12} g_{53} = -b_{12} g_{55}, \quad a_{21} g_{53} = -b_{21} g_{55}.$$

The products $e_1 e_3^2$, $e_2 e_3^2$, $e_3^2 e_1$, $e_3^2 e_2$ give

$$(119) \quad b_{13} g_{53} = b_{23} g_{53} = b_{31} g_{35} = b_{32} g_{35} = 0.$$

Using $e_3 e_1 e_3$ we have $b_{13} g_{35} = b_{31} g_{53}$. Hence $b_{13} b_{31} g_{35} = b_{31}^2 g_{53} = 0$ by (119), $b_{31} g_{53} = 0$. From $e_3 e_2 e_3$ we obtain $b_{23} g_{35} = b_{32} g_{53}$; then $b_{23} b_{32} g_{35} = b_{32}^2 g_{53} = 0$. Similarly we find $b_{13} g_{35} = 0 = b_{23} g_{35}$. These relations together with (119) may be written as

$$(120) \quad b_{13} g_{53} = b_{31} g_{53} = b_{23} g_{53} = b_{32} g_{53} = 0$$

and

$$(121) \quad b_{13} g_{35} = b_{31} g_{35} = b_{23} g_{35} = b_{32} g_{35} = 0.$$

Now suppose that $e_5^2 \neq 0$ in algebras (115). Then by (117) $b_{13} = b_{31} = b_{23} = b_{32} = 0$ and in view of (116), $d_{31} = 0$. Now put $e_3' = e_3 - g_{55}^{-1} g_{53} e_5$ and obtain $e_5 e_3 = 0$. The algebras (115)

may now be written in the form

$$\begin{aligned}
 (122) \quad & e_1^2 = a_{11}e_3 + b_{11}e_5 + d_{11}e_{n-1} + g_{11}e_n, \\
 & e_1e_2 = a_{12}e_3 + b_{12}e_5, \quad e_1e_3 = g_{13}e_n, \\
 & e_2e_1 = a_{21}e_3 + b_{21}e_5 + d_{21}e_{n-1} + g_{21}e_n, \quad e_3e_1 = g_{31}e_n, \\
 & e_2e_3 = g_{23}e_n, \quad e_4e_1 = g_{41}e_n, \quad e_1e_5 = g_{15}e_n, \quad e_5e_1 = g_{51}e_n, \\
 & e_2e_5 = g_{25}e_n, \quad e_3e_5 = g_{35}e_n, \quad e_3^2 = g_{33}e_n, \quad e_5^2 = g_{55}e_n, \\
 & e_2^2 = e_4, \quad e_2^{1-3} = e_1 \quad (i = 6, \dots, n).
 \end{aligned}$$

From $e_5e_1^2$, $e_5e_1e_2$, $e_5e_2e_1$ we find that in (122) $b_{11}e_5^2 = b_{12}e_5^2 = b_{21}e_5^2 = 0$ and since $e_5^2 \neq 0$, we have

$$(123) \quad b_{11} = b_{12} = b_{21} = 0.$$

The products $e_1^2e_2$, $e_2e_1e_2$, $e_2^2e_1$ give $d_{11} = a_{12}g_{13}$, $d_{21} = a_{12}g_{23}$, $g_{41} = g_{23}(a_{12} + a_{21})$. In view of these relations and (123) we may write (122) in the form

$$\begin{aligned}
 (124) \quad & e_1^2 = a_{11}e_3 + a_{12}g_{13}e_{n-1} + g_{11}e_n, \\
 & e_1e_2 = a_{12}e_3, \quad e_2e_1 = a_{21}e_3 + a_{12}g_{23}e_{n-1} + g_{21}e_n, \\
 & e_1e_3 = g_{13}e_n, \quad e_3e_1 = g_{31}e_n, \quad e_2e_3 = g_{23}e_n, \\
 & e_4e_1 = g_{23}(a_{12} + a_{21})e_n, \quad e_1e_5 = g_{15}e_n, \quad e_5e_1 = g_{51}e_n, \\
 & e_2e_5 = g_{25}e_n, \quad e_3e_5 = g_{35}e_n, \quad e_3^2 = g_{33}e_n, \\
 & e_5^2 = g_{55}e_n, \quad e_2^2 = e_4, \quad e_2^{1-3} = e_1 \quad (i = 6, \dots, n).
 \end{aligned}$$

Just as in Chapter IV, Section 3, we have

$$(77) \quad a_{11}e_3^2 = a_{12}e_3^2 = a_{21}e_3^2 = 0.$$

Now let $e_3^2 \neq 0$ in (124). Then by (77), $a_{11} = a_{12} = a_{21} = 0$. Then under the cyclic permutation $(e_2e_4e_5)$ of basal units (124) becomes a regular algebra which may be reduced to the form of algebra 8 or of algebra 9 in n units.

Suppose that $e_3^2 = 0$ in (124). Then after the cyclic permutation $(e_1e_2e_3e_4e_5)$ of basal units the algebra is either algebra 10 or algebra 11.

We have shown that if $e_5^2 \neq 0$ in (115), the algebras are reduced to the form of one of algebras 8, 9, 10, 11. Hence we now assume that $e_5^2 = 0$. Suppose that also $e_5e_3 \neq 0$. Then by (118) and (120) we have $a_{11} = a_{12} = a_{21} = b_{13} = b_{31} = b_{23} = b_{32} = 0$. Then the products $e_1^2e_3$, $e_1e_2e_3$, $e_2e_1e_3$ give $b_{11} = b_{12} = b_{21} = 0$ since $e_5e_3 \neq 0$. Now the cyclic permutation $(e_2e_4e_5)$ transforms the algebra into a regular algebra which by the method used in Chapter III, may be reduced to the form of algebra 8 or of algebra 9. If we assume $e_3e_5 \neq 0$ instead of $e_5e_3 \neq 0$, we obtain the same result; for it is easy to show that (118) holds if g_{53} is replaced by g_{35} ; then we use (121) instead of (120); and finally use the products $e_3e_1^2$, $e_3e_1e_2$, $e_3e_2e_1$ to prove that $b_{11} = b_{12} = b_{21} = 0$.

Now suppose that $e_5^2 = e_5e_3 = e_3e_5 = 0$ in (115). In the commutative case we have algebra 12; in the non-commutative, algebra 13, in n units ($n > 7$). The cases in which $n \leq 7$ will be considered later. If $a_{11} = a_{12} = a_{21} = 0$, the algebras take the form

$$\begin{aligned}
 e_1^2 &= b_{11}e_5 + b_{12}g_{15}e_{n-1} + g_{11}e_n, \\
 e_1e_2 &= b_{12}e_5, & e_2e_1 &= b_{21}e_5 + b_{12}g_{25}e_{n-1} + g_{21}e_n, \\
 e_1e_3 &= b_{13}e_5 + b_{32}g_{15}e_{n-1} + g_{13}e_n, & e_3e_1 &= b_{31}e_5 + g_{31}e_n, \\
 (125) \quad e_2e_3 &= b_{23}e_5 + b_{32}g_{25}e_{n-1} + g_{23}e_n, & e_3e_2 &= b_{32}e_5, \\
 e_4e_1 &= g_{25}(b_{12} + b_{21})e_n, & e_1e_5 &= g_{15}e_n, & e_5e_1 &= g_{51}e_n, \\
 e_2e_5 &= g_{25}e_n, & e_4e_3 &= g_{25}(b_{23} + b_{32})e_n, \\
 e_3^2 &= g_{33}e_n, & e_2^2 &= e_4, & e_2^{1-3} &= e_1 \quad (i = 6, \dots, n).
 \end{aligned}$$

If $b_{32} = 0$, we subject the basal units to the cyclic permutations $(e_1 e_2 e_3)$ and $(e_4 e_5)$ and obtain algebra 10 or algebra 11.

Suppose now $b_{32} \neq 0$ and put $e_1' = e_1 - b_{32}^{-1} b_{12} e_3$. This transformation gives $e_1 e_2 = 0$, eliminates the coefficients of e_{n-1} in the products e_1^2 and $e_2 e_1$ and gives a new g_{41} . The product $e_2^2 e_1$ gives $g_{41} = b_{21} g_{25}$. We now write (125) as

$$\begin{aligned}
 (126) \quad e_1^2 &= b_{11} e_5 + g_{11} e_n, & e_2 e_1 &= b_{21} e_5 + g_{21} e_n, \\
 e_1 e_3 &= b_{13} e_5 + b_{32} g_{15} e_{n-1} + g_{13} e_n, & e_3 e_1 &= b_{31} e_5 + g_{31} e_n, \\
 e_2 e_3 &= b_{23} e_5 + b_{32} g_{25} e_{n-1} + g_{23} e_n, & e_3 e_2 &= b_{32} e_5, \\
 e_4 e_1 &= b_{21} g_{25} e_n, & e_1 e_5 &= g_{15} e_n, & e_5 e_1 &= g_{51} e_n, \\
 e_2 e_5 &= g_{25} e_n, & e_4 e_3 &= g_{25} (b_{23} + b_{32}) e_n, & e_3^2 &= g_{33} e_n, \\
 e_2^2 &= e_4, & e_2^{1-3} &= e_1 & (1 = 6, \dots, n).
 \end{aligned}$$

We next interchange e_2 and e_3 , e_4 and e_5 in (126) and again obtain algebra 10 or algebra 11. Since our algebras are always reduced to algebra 10 or 11 if $a_{11} = a_{12} = a_{21} = 0$, we assume in the final classification that in algebras 12 and 13 in n units a, d, g are not all zero.

7. Algebras derived from algebras 12 and 13 in $n - 1$ units by means of Theorem 8₁ ($n > 8$). We write the new algebras in the form

$$\begin{aligned}
 (127) \quad e_1^2 &= a_{11} e_3 + b_{11} e_5 + c_{11} e_{n-3} + d_{11} e_{n-2} + g_{11} e_{n-1} + h_{11} e_n, \\
 e_1 e_2 &= a_{12} e_3 + b_{12} e_5 + h_{12} e_n, \\
 e_2 e_1 &= a_{21} e_3 + b_{21} e_5 + c_{21} e_{n-3} + d_{21} e_{n-2} + g_{21} e_{n-1} + h_{21} e_n, \\
 e_1 e_3 &= b_{13} e_5 + d_{13} e_{n-2} + g_{13} e_{n-1} + h_{13} e_n, \\
 e_3 e_1 &= b_{31} e_5 + d_{31} e_{n-2} + g_{31} e_{n-1} + h_{31} e_n, \\
 e_2 e_3 &= b_{23} e_5 + d_{23} e_{n-2} + g_{23} e_{n-1} + h_{23} e_n, \\
 e_3 e_2 &= b_{32} e_5 + h_{32} e_n, & e_1 e_4 &= b_{14} e_5 + h_{14} e_n,
 \end{aligned}$$

$$\begin{aligned}
c_4e_1 &= b_{41}e_5 + d_{41}e_{n-2} + g_{41}e_{n-1} + h_{41}e_n, \\
e_1e_5 &= g_{15}e_{n-1} + h_{15}e_n, & e_5e_1 &= g_{51}e_{n-1} + h_{51}e_n, \\
e_1e_6 &= h_{16}e_n, & e_6e_1 &= g_{61}e_{n-1} + h_{61}e_n, \\
e_2e_5 &= g_{25}e_{n-1} + h_{25}e_n, & e_5e_2 &= h_{52}e_n, \\
(127) \quad e_3e_4 &= h_{34}e_n, & e_4e_3 &= g_{43}e_{n-1} + h_{43}e_n, & e_3^2 &= g_{33}e_{n-1} + h_{33}e_n, \\
e_1e_i &= h_{1i}e_n, & e_ie_1 &= h_{i1}e_n & (i = 7, \dots, n-1), \\
e_3e_i &= h_{3i}e_n, & e_ie_3 &= h_{i3}e_n & (i = 5, \dots, n-1), \\
e_5e_i &= h_{5i}e_n, & e_ie_5 &= h_{i5}e_n & (i = 4, \dots, n-1), \\
e_2^2 &= e_4, & e_2^{1-3} &= e_1 & (i = 6, \dots, n).
\end{aligned}$$

By the associative law $e_2e_5e_2 = 0 = g_{25}e_{n-1}e_2 = g_{25}e_n$ and $g_{25} = 0$. Then from $e_1e_5e_2$ and $e_2e_5e_1$ we find $g_{15} = 0 = g_{51}$. We also have $e_5e_1 = 0 = e_1e_5$ ($i = 4, 6, \dots, n-1$). It follows that $e_3e_2e_4 = 0 = e_3e_6$. Also $e_2e_3e_4 = 0 = d_{23}e_{n-2}e_4 = d_{23}e_n$ and $d_{23} = 0$, $e_4e_2e_3 = 0 = e_6e_3$. It then follows that $e_3e_1 = 0 = e_1e_3$ ($i = 6, \dots, n-1$), $e_6e_1e_2 = 0 = g_{61}e_{n-1}e_2 = g_{61}e_n$ and $g_{61} = 0$, $e_6e_1^2 = 0 = c_{11}e_6e_{n-3} = c_{11}e_n$, $c_{11} = 0$. Then $e_1e_i = 0 = e_ie_1$ ($i = 7, \dots, n-1$), so that (113) holds. From the products $e_6e_2e_1$, $e_1e_3e_4$, $e_2e_4e_3$, $e_4e_3e_1$, $e_4^2e_1$ we find $c_{21} = d_{13} = g_{43} = d_{31} = d_{41} = 0$. If now $g_{33} = 0$, the algebras have the form (112) together with (113) and we finally have algebras 12 and 13 in n units.

We now assume that $g_{33} \neq 0$. We apply to the algebras the transformation (114) with the g_{ij} replaced by h_{ij} . The coefficients of e_n in the products e_1e_2 , e_3e_2 , e_5e_2 are eliminated and the algebras take the form

$$\begin{aligned}
e_1^2 &= a_{11}e_3 + b_{11}e_5 + d_{11}e_{n-2} + g_{11}e_{n-1} + h_{11}e_n, \\
e_1e_2 &= a_{12}e_3 + b_{12}e_5, \\
(128) \quad e_2e_1 &= a_{21}e_3 + b_{21}e_5 + d_{21}e_{n-2} + g_{21}e_{n-1} + h_{21}e_n, \\
e_1e_3 &= b_{13}e_5 + g_{13}e_{n-1} + h_{13}e_n, \\
e_3e_1 &= b_{31}e_5 + g_{31}e_{n-1} + h_{31}e_n,
\end{aligned}$$

$$\begin{aligned}
 e_2 e_3 &= b_{23} e_5 + g_{23} e_{n-1} + h_{23} e_n, & e_3 e_2 &= b_{32} e_5, \\
 e_1 e_4 &= b_{14} e_5 + g_{14} e_{n-1} + h_{14} e_n, & e_1 e_5 &= h_{15} e_n, \\
 e_4 e_1 &= b_{41} e_5 + g_{41} e_{n-1} + h_{41} e_n, & e_5 e_1 &= h_{51} e_n, \\
 (128) \quad e_1 e_6 &= h_{16} e_n, \quad e_6 e_1 = h_{61} e_n, \quad e_2 e_5 = h_{25} e_n, \\
 e_3 e_4 &= h_{34} e_n, \quad e_4 e_3 = h_{43} e_n, \quad e_3 e_5 = h_{35} e_n, \quad e_5 e_3 = h_{53} e_n, \\
 e_3^2 &= g_{33} e_{n-1} + h_{33} e_n, & e_5^2 &= h_{55} e_n, \\
 e_2^2 &= e_4, \quad e_2^{1-3} = e_1 & (i = 6, \dots, n).
 \end{aligned}$$

The associative law gives $e_3 e_2^2 = 0 = e_3 e_4$, $e_1 e_2 e_4 = 0 = e_1 e_6$. From $e_1 e_2^2$, we have $g_{14} = 0 = h_{14}$, $b_{14} = a_{12} b_{32}$. The products $e_3 e_1^2$, $e_3 e_1 e_2$, $e_3 e_2 e_1$ give $a_{11} g_{33} = a_{12} g_{33} = a_{21} g_{33} = 0$ and therefore $a_{11} = a_{12} = a_{21} = 0$ since $g_{33} \neq 0$. From $e_4 e_1 e_2$ we have $g_{41} = 0$; from $e_2^2 e_1$, $b_{41} = 0 = d_{21}$. Using $e_2 e_3^2$ and $e_3^2 e_2$ we find $g_{33} = b_{23} h_{53} = b_{32} h_{35}$ and therefore b_{23} , h_{53} , b_{32} , h_{35} are all different from zero. Therefore from $e_5 e_2 e_3$ we find $e_5^2 = 0$, and from $e_1 e_3^2$, $b_{13} = 0$. From $e_4 e_1^2$, $e_2 e_4 e_1$, $e_3^2 e_1$ we have $d_{11} = h_{61} = b_{31} = 0$. Now put $e_1' = e_1 - h_{35}^{-1} h_{15} e_3$ and obtain $e_1 e_5 = 0$. Then $e_1 e_2 e_3 = 0 = b_{12} e_5 e_3$ and $b_{12} = 0$ since $h_{53} \neq 0$. The products $e_1^2 e_2$, $e_2 e_1 e_2$, $e_1 e_3 e_2$, $e_3 e_1 e_2$ give $g_{11} = g_{21} = g_{13} = g_{31} = 0$. From $e_2 e_3 e_2$, $e_2^2 e_1$, $e_2^2 e_3$, $e_3^2 e_2$ we have $g_{23} = b_{32} h_{25}$, $h_{41} = b_{21} h_{25}$, $h_{43} = h_{25} (b_{23} + b_{32})$, $g_{33} = b_{32} g_{35}$. If now we interchange e_2 and e_3 , e_4 and e_5 , our algebras become algebra 10 or algebra 11.

We have now shown that the algebras 12 and 13 in $n - 1$ units give rise to 12 and 13 or 10 and 11 in n units by the use of Theorem 8₁. But in order to make our induction complete and to show that algebras 12 and 13 are perfectly general, we must examine a few special cases, since the multiplication tables found for 12 and 13 hold only for $n > 7$.

8. Algebras of order 5 derived from algebras 6 and 7 in 4 units by means of Theorem 7₂ with $s = n - 1$. The new algebras take the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5, \\
 e_1e_2 &= a_{12}e_3 + c_{12}e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5, \\
 (129) \quad e_1e_3 &= c_{13}e_5, & e_3e_1 &= c_{31}e_5, & e_1e_4 &= c_{14}e_5, & e_4e_1 &= c_{41}e_5, \\
 e_2e_3 &= c_{23}e_5, & e_3e_2 &= c_{32}e_5, & e_3e_4 &= c_{34}e_5, & e_4e_3 &= c_{43}e_5, \\
 e_3^2 &= c_{33}e_5, & e_2^2 &= e_4.
 \end{aligned}$$

By the associative law $e_2^2e_3 = 0 = e_4e_3$, $e_3e_2^2 = 0 = e_3e_4$. From $e_1e_2^2$ and $e_2^2e_1$ we obtain $c_{14} = a_{12}c_{32}$, $c_{41} = a_{21}c_{23}$. As in Chapter IV, Section 3, we have

$$(77) \quad a_{11}e_3^2 = a_{12}e_3^2 = a_{21}e_3^2 = 0.$$

If $e_3^2 \neq 0$ then $a_{11} = a_{12} = a_{21} = 0$ and (129) becomes

$$\begin{aligned}
 e_1^2 &= b_{11}e_4 + c_{11}e_5, & e_1e_2 &= c_{12}e_5, & e_2e_1 &= b_{21}e_4 + c_{21}e_5, \\
 (130) \quad e_1e_3 &= c_{13}e_5, & e_3e_1 &= c_{31}e_5, & e_2e_3 &= c_{23}e_5, & e_3e_2 &= c_{32}e_5, \\
 e_3^2 &= c_{33}e_5, & e_2^2 &= e_4.
 \end{aligned}$$

After the cyclic permutation $(e_1e_2e_3)$ of basal units (130) is a case of the algebras (88), from which (90) and (91) are obtained, with $n = 5$. Hence we assume $e_3^2 = 0$ and a_{11} , a_{12} , a_{21} not all zero. In the commutative case we have algebra 12 and in the non-commutative case, algebra 13, of order 5.

9. Algebras of order 6 arising from algebras 12 and 13 in 5 units by the use of Theorem 7₁ with $s = 4$. The new algebras have the multiplication tables

$$\begin{aligned}
e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6, \\
e_1e_2 &= a_{12}e_3 + c_{12}e_5 + d_{12}e_6, & e_1e_3 &= c_{13}e_5 + d_{13}e_6, \\
e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6, & e_3e_1 &= c_{31}e_5 + d_{31}e_6, \\
e_2e_3 &= c_{23}e_5 + d_{23}e_6, & e_1e_4 &= c_{14}e_5 + d_{14}e_6, & e_3e_4 &= d_{34}e_6, \\
e_3e_2 &= c_{32}e_5 + d_{32}e_6, & e_4e_1 &= c_{41}e_5 + d_{41}e_6, & e_4e_3 &= d_{43}e_6, \\
e_1e_5 &= d_{15}e_6, & e_5e_1 &= d_{51}e_6 & (1 = 1, 2, 3, 4, 5), \\
e_3^2 &= d_{33}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6.
\end{aligned}$$

By the associative law $e_2^2e_5 = 0 = e_4e_5$, $e_5e_2^2 = 0 = e_5e_4$. Now put $e_1' = e_1 - a_{12}d_{32}e_2 - d_{12}e_4$, $e_3' = e_3 - d_{32}e_4$. This transformation eliminates the coefficients of e_6 in the products e_1e_2 and e_3e_2 . The algebras now have the form

$$\begin{aligned}
e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6, \\
e_1e_2 &= a_{12}e_3 + c_{12}e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6, \\
e_1e_3 &= c_{13}e_5 + d_{31}e_6, & e_3e_1 &= c_{31}e_5 + d_{31}e_6, \\
(131) \quad e_2e_3 &= c_{23}e_5 + d_{23}e_6, & e_3e_2 &= c_{32}e_5, & e_1e_4 &= c_{14}e_5 + d_{14}e_6, \\
e_4e_1 &= c_{41}e_5 + d_{41}e_6, & e_3e_4 &= d_{34}e_6, & e_4e_3 &= d_{43}e_6, \\
e_1e_5 &= d_{15}e_6, & e_5e_1 &= d_{51}e_6 & (1 = 1, 2, 3, 5), \\
e_3^2 &= d_{33}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6.
\end{aligned}$$

Applying the associative law to the products $e_5e_1e_3$, $e_5e_3e_1$, $e_5e_2e_3$, $e_5e_3e_2$, we have

$$(132) \quad c_{13}e_5^2 = c_{31}e_5^2 = c_{23}e_5^2 = c_{32}e_5^2 = 0.$$

Using the products $e_3e_1e_3$, $e_3e_2e_3$, $e_1e_3^2$, $e_2e_3^2$, $e_3^2e_1$, $e_3^2e_2$ and the method used on page 45, we have

$$\begin{aligned}
(133) \quad c_{13}d_{53} &= c_{31}d_{53} = c_{23}d_{53} = c_{32}d_{53} = c_{13}d_{35} \\
&= c_{31}d_{35} = c_{23}d_{35} = c_{32}d_{35} = 0.
\end{aligned}$$

From the products $e_5e_1^2$, $e_1^2e_5$, $e_5e_1e_2$, $e_1e_2e_5$, $e_5e_2e_1$, $e_2e_1e_5$

we obtain

$$(134) \quad \begin{aligned} a_{11}d_{53} &= -c_{11}d_{55} = a_{11}d_{35}, \\ a_{12}d_{53} &= -c_{12}d_{55} = a_{12}d_{35}, \quad a_{21}d_{53} = -c_{21}d_{55} = a_{21}d_{35}. \end{aligned}$$

We first consider the case $e_5^2 \neq 0$ in (131). Then by (132), $c_{13} = c_{31} = c_{23} = c_{32} = 0$. Hence $e_3e_2^2 = 0 = e_3e_4$, $e_2^2e_3 = 0 = e_4e_3$. From $e_1e_2^2$ and $e_2^2e_1$ we have $c_{14} = 0 = c_{41}$, $d_{14} = c_{12}d_{52}$. Now put $e_1' = e_1 - c_{12}d_{52}e_2$, $e_3' = e_3 + d_{55}^{-1}d_{52}d_{53}e_4 - d_{55}^{-1}d_{53}e_5$, $e_5' = -d_{52}e_4 + e_5$, and obtain $e_1e_4 = 0 = e_5e_2 = e_5e_3$. From the products $e_5e_1^2$, $e_5e_1e_2$, $e_5e_2e_1$, we have $c_{11} = c_{12} = c_{21} = 0$ since $e_5^2 \neq 0$. Also from $e_1^2e_2$, $e_2e_1e_2$, $e_2^2e_1$ we find $b_{11} = a_{12}d_{13}$, $b_{21} = a_{12}d_{23}$, $d_{41} = d_{23}(a_{12} + a_{21})$. Our algebras are now in the form (124) with the g_{ij} replaced by d_{ij} , e_{n-1} by e_4 and e_n by e_5 . With these replacements the remainder of the discussion of (124) on page 46 holds here also and we have algebras 8 and 9 or algebras 10 and 11 for $n = 6$.

Now we assume that $e_5^2 = 0$ in (131). Suppose also that $e_5e_3 \neq 0$ or $e_3e_5 \neq 0$. Then by (133) and (134) we have $c_{13} = c_{31} = c_{23} = c_{32} = a_{11} = a_{12} = a_{21} = 0$. Also $e_3e_2^2 = 0 = e_3e_4$, $e_2^2e_3 = 0 = e_4e_3$, and from $e_1e_2^2$, $e_2^2e_1$ we find $c_{14} = a_{12}c_{32} = 0$, $c_{41} = a_{21}c_{23} = 0$. If $e_5e_3 \neq 0$ the products $e_1^2e_3$, $e_1e_2e_3$, $e_2e_1e_3$ give $c_{11} = c_{12} = c_{21} = 0$, while for $e_3e_5 \neq 0$ we obtain the same result from $e_3e_1^2$, $e_3e_1e_2$, $e_3e_2e_1$. From $e_1^2e_2$ and $e_2e_1e_2$ we find $b_{11} = 0 = b_{21}$. After the cyclic permutation $(e_2e_4e_5)$ of the basal units we have a regular algebra which takes the form of algebra 8 or algebra 9 for $n = 6$ under the transformations of Chapter III.

We next assume $e_5^2 = e_3e_5 = e_5e_3 = 0$ in (131). Now put $e_1' = e_1 - c_{12}d_{52}e_2$, $e_5' = -d_{52}e_4 + e_5$. This transformation eliminates e_5e_2 but introduces some terms in e_4 . The algebras

assume the form¹

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6, & e_2^2 &= e_4, \\
 e_1e_2 &= a_{12}e_3 + c_{12}e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6, \\
 e_1e_3 &= b_{13}e_4 + c_{13}e_5 + d_{13}e_6, & e_3e_1 &= b_{31}e_4 + c_{31}e_5 + d_{31}e_6, \\
 (135) \quad e_2e_3 &= b_{23}e_4 + c_{23}e_5 + d_{23}e_6, & e_3e_2 &= b_{32}e_4 + c_{32}e_5, \\
 e_1e_4 &= c_{14}e_5 + d_{14}e_6, & e_4e_1 &= b_{41}e_4 + c_{41}e_5 + d_{41}e_6, \\
 e_1e_5 &= d_{15}e_6, & e_5e_1 &= b_{51}e_4 + c_{51}e_5 + d_{51}e_6, & e_2e_5 &= d_{25}e_6, \\
 e_3e_4 &= d_{34}e_6, & e_4e_3 &= d_{43}e_6, & e_3^2 &= d_{33}e_6, & e_2^3 &= e_6.
 \end{aligned}$$

By the associative law $e_5e_1e_2 = 0 = b_{51}e_4e_2 = b_{51}e_6$ and $b_{51} = 0$. Then $e_5e_1^2 = 0 = c_{51}e_5e_1 = c_{51}^2e_5 + c_{51}d_{51}e_6$ and hence $c_{51} = 0$. From $e_3^2e_2$ and $e_3e_2^2$ we have $b_{32}d_{34} = 0$ and $b_{32} = d_{34}$. Therefore $b_{32} = 0 = d_{34}$. The product $e_1e_2^2$ gives $d_{14} = 0$, $c_{14} = a_{12}c_{32}$. Since $e_4e_1^2 = a_{11}e_4e_3 = a_{11}d_{43}e_6 = b_{41}e_4e_1 + c_{41}e_5e_1 = b_{41}^2e_4 + b_{41}c_{41}e_5 + (b_{41}d_{41} + c_{41}d_{51})e_6$, it follows that $b_{41} = 0$. From $e_1e_3e_2$ and $e_3e_1e_2$ we have $b_{13} = c_{32}d_{15}$, $b_{31} = a_{12}d_{33}$. From $e_1^2e_2$ we obtain $b_{11} = a_{12}d_{13} + c_{12}d_{15}$ and

$$(136) \quad a_{12}c_{32}d_{15} = 0 = a_{12}b_{13}.$$

From $e_1e_2e_1$ we find $a_{12}^2d_{33} = a_{21}c_{32}d_{15}$. Therefore $a_{12}^3d_{33} = a_{12}a_{21}c_{32}d_{15} = 0$ by (136). Hence $a_{12}d_{33} = 0 = b_{31}$ and

$$(137) \quad a_{21}c_{32}d_{15} = 0 = a_{21}b_{13}.$$

From $e_2^3e_2$ we have $b_{23} = c_{32}d_{25}$; from $e_2^2e_3$, $d_{43} = d_{25}(c_{23} + c_{32})$; from $e_2e_1e_2$, $b_{21} = a_{12}d_{23} + c_{12}d_{25}$ and

$$(138) \quad a_{12}c_{32}d_{25} = 0 = a_{12}b_{23}.$$

¹In (131) the product $e_1e_4e_2$ gives $c_{14}d_{52} = 0$. Hence in (135) no term in e_4 appears in the products e_1e_4 and e_1e_5 .

From $e_2e_1^2$ and $e_2^2e_1$ we find

$$(139) \quad a_{11}b_{23} = 0 = a_{21}b_{23}$$

and $c_{41} = a_{21}c_{23}$, $d_{41} = d_{23}(a_{12} + a_{21}) + d_{25}(c_{12} + c_{21})$. The associative law applied to e_1^3 gives

$$(140) \quad a_{11}b_{13} = 0.$$

We now write (136), ..., (140) as

$$(141) \quad a_{11}b_{13} = a_{12}b_{13} = a_{21}b_{13} = a_{11}b_{23} = a_{12}b_{23} = a_{21}b_{23} = 0,$$

and write (135) in the form

$$(142) \quad \begin{aligned} e_1^2 &= a_{11}e_3 + (a_{12}d_{13} + c_{12}d_{15})e_4 + c_{11}e_5 + d_{11}e_6, \\ e_1e_2 &= a_{12}e_3 + c_{12}e_5, \\ e_2e_1 &= a_{21}e_3 + (a_{12}d_{23} + c_{12}d_{25})e_4 + c_{21}e_5 + d_{21}e_6, \\ e_1e_3 &= c_{32}d_{15}e_4 + c_{13}e_5 + d_{13}e_6, & e_3e_1 &= c_{31}e_5 + d_{31}e_6, \\ e_2e_3 &= c_{32}d_{25}e_4 + c_{23}e_5 + d_{23}e_6, & e_3e_2 &= c_{32}e_5, \\ e_1e_4 &= a_{12}c_{32}e_5, \\ e_4e_1 &= a_{21}c_{23}e_5 + [d_{23}(a_{12} + a_{21}) + d_{25}(c_{12} + c_{21})]e_6, \\ e_1e_5 &= d_{15}e_6, & e_5e_1 &= d_{51}e_6, & e_2e_5 &= d_{25}e_6, & e_2^2 &= e_4, \\ e_4e_3 &= d_{25}(c_{23} + c_{32})e_6, & e_3^2 &= d_{33}e_6, & e_2^3 &= e_6. \end{aligned}$$

Now consider the case $b_{13} \neq 0$ or $b_{23} \neq 0$, that is, $c_{32}d_{15} \neq 0$ or $c_{32}d_{25} \neq 0$. Then by (141), $a_{11} = a_{12} = a_{21} = 0$. Remembering that $c_{32} \neq 0$, we put

$$(143) \quad e_1' = e_1 - c_{32}^{-1}c_{12}e_3.$$

The algebras (142) now have $e_1e_2 = 0$ and take the form

$$\begin{aligned}
 (144) \quad & e_1^2 = c_{11}e_5 + d_{11}e_6, & e_2e_1 &= c_{21}e_5 + d_{21}e_6, \\
 & e_1e_3 = c_{32}d_{15}e_4 + c_{13}e_5 + d_{13}e_6, & e_3e_1 &= c_{31}e_5 + d_{31}e_6, \\
 & e_2e_3 = c_{32}d_{25}e_4 + c_{23}e_5 + d_{23}e_6, & e_3e_2 &= c_{32}e_5, \\
 & e_4e_1 = c_{21}d_{25}e_6, & e_1e_5 &= d_{15}e_6, & e_5e_1 &= d_{51}e_6, \\
 & e_2e_5 = d_{25}e_6, & e_4e_3 &= d_{25}(c_{23} + c_{32})e_6, \\
 & e_3^2 = d_{33}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6.
 \end{aligned}$$

If in (144) we interchange e_2 and e_3 , and e_4 and e_5 , we have algebra 10 or algebra 11 for $n = 6$. Hence we assume $b_{13} = 0 = b_{23}$ in (142), and now have algebras 12 and 13 for $n = 6$.

If in (142) with $b_{13} = 0 = b_{23}$ we have $a_{11} = a_{12} = a_{21} = 0$, the algebras are always reduced to the form of algebra 10 or algebra 11; for if $c_{32} \neq 0$ the transformation (143) gives (144) with $b_{13} = b_{23} = 0$, a case of algebra 10 or algebra 11, while if $c_{32} = 0$, the cyclic permutations $(e_1e_2e_3)$ and (e_4e_5) transform the algebras into 10 or 11. Hence in the final classification we assume a, d, g not all zero in algebras 12 and 13 for $n = 6$.

10. Algebras of order 7 arising from algebras 12 and 13 in 6 units by the use of Theorem 8₁. The new algebras take the form

$$\begin{aligned}
 (145) \quad & e_1^2 = a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6 + g_{11}e_7, \\
 & e_1e_2 = a_{12}e_3 + c_{12}e_5 + g_{12}e_7, \\
 & e_2e_1 = a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7, \\
 & e_1e_3 = c_{13}e_5 + d_{13}e_6 + g_{13}e_7, & e_3e_1 &= c_{31}e_5 + d_{31}e_6 + g_{31}e_7, \\
 & e_2e_3 = c_{23}e_5 + d_{23}e_6 + g_{23}e_7, & e_3e_2 &= c_{32}e_5 + g_{32}e_7, \\
 & e_1e_4 = c_{14}e_5 + g_{14}e_7, & e_4e_1 &= c_{41}e_5 + d_{41}e_6 + g_{41}e_7, \\
 & e_1e_5 = d_{15}e_6 + g_{15}e_7, & e_5e_1 &= d_{51}e_6 + g_{51}e_7, \\
 & e_2e_5 = d_{25}e_6 + g_{25}e_7, & e_5e_2 &= g_{52}e_7, & e_3e_4 &= g_{34}e_7, \\
 & e_4e_3 = d_{43}e_6 + g_{43}e_7, & e_3^2 &= d_{33}e_6 + g_{33}e_7, \\
 & e_1e_6 = g_{16}e_7, & e_6e_1 &= g_{61}e_7,
 \end{aligned}$$

$$\begin{aligned}
 (145) \quad & e_1 e_5 = g_{15} e_7, & e_5 e_1 = g_{51} e_7 & (1 = 3, 4, 5, 6), \\
 & e_3 e_6 = g_{36} e_7, & e_6 e_3 = g_{63} e_7, \\
 & e_2^2 = e_4, & e_2^3 = e_6, & e_2^4 = e_7.
 \end{aligned}$$

The associative law applied to the products $e_2 e_5 e_2$, $e_1 e_5 e_2$, $e_2 e_5 e_1$ gives $d_{25} = d_{15} = d_{51} = 0$. Then $e_2^2 e_5 = 0 = e_4 e_5$, $e_5 e_2^2 = 0 = e_5 e_4$, $e_2^3 e_5 = 0 = e_6 e_5$, $e_5 e_2^3 = 0 = e_5 e_6$. From $e_4 e_3 e_2$ we have $d_{43} = 0$. Then $e_2 e_4 e_3 = 0 = e_6 e_3$, $e_3 e_4 e_2 = 0 = e_3 e_6$. From $e_1 e_2^2$ we find

$$(146) \quad c_{14} = a_{12} c_{32}, \quad g_{14} = a_{12} g_{32} + c_{12} g_{52}.$$

Now put $e_1' = e_1 - a_{12} c_{32} g_{52} e_2 - (a_{12} g_{32} + c_{12} g_{52}) e_4 - g_{12} e_6$, $e_3' = e_3 - c_{32} g_{52} e_4 - g_{32} e_6$, $e_5' = e_5 - g_{52} e_6$, keeping (146) in mind. The coefficients of e_7 in the products $e_1 e_2$, $e_3 e_2$, $e_1 e_4$, $e_5 e_2$ are eliminated. We therefore have $e_3 e_2^2 = 0 = e_3 e_4$, $e_1 e_2 e_4 = 0 = e_1 e_6$. The products $e_1 e_2^2$, $e_1 e_3 e_2$, $e_3 e_1 e_2$ give $c_{14} = a_{12} c_{32}$, $d_{13} = c_{32} g_{15}$, $d_{31} = a_{12} g_{33} + c_{12} g_{35}$ and

$$(147) \quad a_{12} d_{33} = 0.$$

From $e_2 e_3 e_2$, $e_1^2 e_2$ we have $d_{23} = c_{32} g_{25}$, $b_{11} = a_{12} c_{32} g_{15}$, $d_{11} = a_{12} g_{13} + c_{12} g_{15}$. From $e_2 e_1 e_2$ and $e_2^2 e_3$ we find $b_{21} = a_{12} c_{32} g_{25}$, $d_{21} = a_{12} g_{23} + c_{12} g_{25}$, $g_{43} = g_{25}(c_{23} + c_{32})$. The products $e_4 e_2 e_1$, $e_2^2 e_1$ give $g_{61} = g_{25}(a_{21} c_{23} + a_{21} c_{32} + a_{12} c_{32})$, $c_{41} = a_{21} c_{23}$, $d_{41} = c_{32} g_{25}(a_{12} + a_{21})$, $g_{41} = g_{23}(a_{12} + a_{21}) + g_{25}(c_{12} + c_{21})$. We may now write the algebras in the form¹

¹The product $e_2 e_3 e_1$ gives a second expression for d_{31} in $e_3 e_1$, viz., $c_{23} g_{51} - c_{31} g_{25} + c_{32} g_{25} g_{61}$.

$$\begin{aligned}
e_1^2 &= a_{11}e_3 + a_{12}c_{32}g_{15}e_4 + c_{11}e_5 \\
&\quad + (a_{12}g_{13} + c_{12}g_{15})e_6 + g_{11}e_7, \\
e_1e_2 &= a_{12}e_3 + c_{12}e_5, \quad e_2e_1 = a_{21}e_3 + a_{12}c_{32}g_{25}e_4 + c_{21}e_5 \\
&\quad + (a_{12}g_{23} + c_{12}g_{25})e_6 + g_{21}e_7, \\
e_1e_3 &= c_{13}e_5 + c_{32}g_{15}e_6 + g_{13}e_7, \\
e_3e_1 &= c_{31}e_5 + (a_{12}g_{33} + c_{12}g_{35})e_6 + g_{31}e_7, \\
(148) \quad e_2e_3 &= c_{23}e_5 + c_{32}g_{25}e_6 + g_{23}e_7, \quad e_3e_2 = c_{32}e_5, \\
e_1e_4 &= a_{12}c_{32}e_5, \quad e_4e_1 = a_{21}c_{23}e_5 + c_{32}g_{25}(a_{12} + a_{21})e_6 \\
&\quad + [g_{23}(a_{12} + a_{21}) + g_{25}(c_{12} + c_{21})]e_7, \\
e_1e_5 &= g_{15}e_7, \quad e_5e_1 = g_{51}e_7, \quad e_3e_5 = g_{35}e_7, \quad e_5e_3 = g_{53}e_7, \\
e_2e_5 &= g_{25}e_7, \quad e_4e_3 = g_{25}(c_{23} + c_{32})e_7, \\
e_6e_1 &= g_{25}(a_{21}c_{23} + a_{21}c_{32} + a_{12}c_{32})e_7, \quad e_5^2 = g_{55}e_7, \\
e_3^2 &= d_{33}e_6 + g_{33}e_7, \quad e_2^2 = e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7.
\end{aligned}$$

Now suppose $d_{33} \neq 0$. The products $e_3e_1^2$, $e_3e_2e_1$ give $a_{11}d_{33} = 0 = a_{21}d_{33}$. These relations with (147) imply that $a_{11} = a_{12} = a_{21} = 0$. From $e_2e_3^2$ and $e_3^2e_2$ we have $d_{33} = c_{23}g_{53} = c_{32}g_{35}$, and therefore c_{23} , c_{32} , g_{35} , g_{53} are all different from zero. Hence the products $e_5e_2e_3$, $e_1e_3^2$, $e_3^2e_1$ give $g_{55} = 0 = c_{13} = c_{31}$. We now put $e_1' = e_1 - g_{35}^{-1}g_{15}e_3$ and eliminate e_1e_5 and the coefficients of e_6 in the products $e_1'^2$, $e_1'e_3$. Since $g_{53} \neq 0$, the product $e_1'e_2e_3$ gives $c_{12} = 0$. Then from $e_2e_1e_2$ and $e_3e_1e_2$ we have $d_{21} = 0 = d_{31}$. From $e_2^2e_1$ we find the new g_{41} to be $c_{21}g_{25}$. The algebras now may be written in the form

$$\begin{aligned}
e_1'^2 &= c_{11}e_5 + g_{11}e_7, \quad e_2e_1 = c_{21}e_5 + g_{21}e_7, \\
e_1'e_3 &= g_{13}e_7, \quad e_3e_1 = g_{31}e_7, \quad e_4e_1 = c_{21}g_{25}e_7, \\
(149) \quad e_2e_3 &= c_{23}e_5 + c_{32}g_{25}e_6 + g_{23}e_7, \quad e_3e_2 = c_{32}e_5, \\
e_5e_1 &= g_{51}e_7, \quad e_2e_5 = g_{25}e_7, \quad e_3e_5 = g_{35}e_7, \quad e_5e_3 = g_{53}e_7, \\
e_4e_3 &= g_{25}(c_{23} + c_{32})e_7, \quad e_3^2 = c_{32}g_{35}e_6 + g_{33}e_7, \\
e_2^2 &= e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7.
\end{aligned}$$

If we interchange e_2 and e_3 and also e_4 and e_5 the algebras (149) become algebra 10 or algebra 11 for $n = 7$.

We now assume $d_{33} = 0$ in (148). Just as in Section 9 we have

$$(132) \quad c_{13}e_5^2 = c_{31}e_5^2 = c_{23}e_5^2 = c_{32}e_5^2 = 0$$

and (133), (134) with d_{1j} replaced by g_{1j} , that is,

$$(150) \quad \begin{aligned} c_{13}g_{53} &= c_{31}g_{53} = c_{23}g_{53} = c_{32}g_{53} = c_{13}g_{35} \\ &= c_{31}g_{35} = c_{23}g_{35} = c_{32}g_{35} = 0, \end{aligned}$$

$$(151) \quad \begin{aligned} a_{11}g_{53} &= -c_{11}g_{55} = a_{11}g_{35}, & a_{12}g_{53} &= -c_{12}g_{55} = a_{12}g_{35}, \\ a_{21}g_{53} &= -c_{21}g_{55} = a_{21}g_{35}. \end{aligned}$$

If $e_5^2 \neq 0$ the algebras (148) are reduced to one of algebras 8, 9, 10, 11. For by (132) $c_{13} = c_{31} = c_{23} = c_{32} = 0$. Hence in view of the footnote on page 57, $d_{31} = 0$. Then the transformation $e_3' = e_3 - g_{55}^{-1}g_{53}e_5$ eliminates e_5e_3 and reduces the algebras to the form (122) with $n = 7$ and the b_{1j} replaced by c_{1j} . The discussion on page 46 of (122) and (124), with these replacements, holds in the present case. We therefore assume $e_5^2 = 0$. If also we have $e_5e_3 \neq 0$ or $e_3e_5 \neq 0$, then by (150) and (151) $c_{13} = c_{31} = c_{23} = c_{32} = a_{11} = a_{12} = a_{21} = 0$, and hence the products $e_1^2e_3$, $e_1e_2e_3$, $e_2e_1e_3$, $e_3e_1^2$, $e_3e_1e_2$, $e_3e_2e_1$ give $c_{11} = c_{12} = c_{21}$. The cyclic permutation $(e_2e_4e_5)$ now transforms the algebras into regular algebras which, under the transformations of Chapter III, assume the form of algebra 8 or of algebra 9. We therefore assume $e_5^2 = e_5e_3 = e_3e_5 = 0 = d_{33}$ in (148) and have algebras 12 and 13 for $n = 7$. Their form is indicated in a note in Chapter VII before algebra 13 in n units.

If in the algebras (148) with $e_5^2 = e_5e_3 = e_3e_5 = 0 = d_{33}$ we have $a_{11} = a_{12} = a_{21} = 0$, we have (125) with the b_{1j} replaced

by c_{1j} and $n = 7$. The discussion on page 48 of (125) holds in the present case and we therefore assume a, d, g not all zero in algebras 12 and 13 in 7 units.

11. Algebras of order 8 derived from algebras 12 and 13 in 7 units by means of Theorem 8₁. The new algebras are

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6 + g_{11}e_7 + h_{11}e_8, \\
 e_1e_2 &= a_{12}e_3 + c_{12}e_5 + h_{12}e_8, \\
 e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7 + h_{21}e_8, \\
 e_1e_3 &= c_{13}e_5 + d_{13}e_6 + g_{13}e_7 + h_{13}e_8, \\
 e_3e_1 &= c_{31}e_5 + d_{31}e_6 + g_{31}e_7 + h_{31}e_8, \\
 e_2e_3 &= c_{23}e_5 + d_{23}e_6 + g_{23}e_7 + h_{23}e_8, \quad e_3e_2 = c_{32}e_5 + h_{32}e_8, \\
 e_1e_4 &= c_{14}e_5 + h_{14}e_8, \quad e_4e_1 = c_{41}e_5 + d_{41}e_6 + g_{41}e_7 + h_{41}e_8, \\
 (152) \quad e_1e_5 &= g_{15}e_7 + h_{15}e_8, \quad e_5e_1 = g_{51}e_7 + h_{51}e_8, \\
 e_2e_5 &= g_{25}e_7 + h_{25}e_8, \quad e_5e_2 = h_{52}e_8, \\
 e_3e_4 &= h_{34}e_8, \quad e_4e_3 = g_{43}e_7 + h_{43}e_8, \\
 e_1e_6 &= h_{16}e_8, \quad e_6e_1 = g_{61}e_7 + h_{61}e_8, \\
 e_1e_7 &= h_{17}e_8, \quad e_7e_1 = h_{71}e_8, \quad e_3^2 = g_{33}e_7 + h_{33}e_8, \\
 e_3e_1 &= h_{31}e_8, \quad e_1e_3 = h_{13}e_8, \quad (i = 5, 6, 7), \\
 e_5e_1 &= h_{51}e_8, \quad e_1e_5 = h_{15}e_8, \quad (i = 4, 5, 6, 7), \\
 e_2^2 &= e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7, \quad e_2^5 = e_8.
 \end{aligned}$$

The associative law applied to the products $e_2e_5e_2$, $e_1e_5e_2$, $e_2e_5e_1$ gives $g_{25} = g_{15} = g_{51} = 0$. It then follows that $e_5e_1 = 0 = e_1e_5$ ($i = 4, 6, 7$). From $e_4e_3e_2$ we find $g_{43} = 0$, and then we have $e_3e_6 = e_6e_3 = e_3e_7 = e_7e_3 = 0$. From $e_6e_1e_2$ we have $g_{61} = 0$ and therefore $e_1e_7 = 0 = e_7e_1$. Using the products $e_1^2e_6$, $e_2e_1e_6$, $e_1e_3e_4$, $e_4e_3e_1$, $e_2e_3e_4$, $e_4e_1e_4$, we find $b_{11} = b_{21} = d_{13} = d_{31} = d_{23} = d_{41} = 0$. If $g_{33} = 0$ the algebras are now in the form (112), (113) with $n = 8$. We therefore obtain algebras 12 and 13 for $n = 8$.

Now suppose that $g_{33} \neq 0$. Put $e_1' = e_1 - a_{12}c_{32}h_{52}e_4 - (a_{12}h_{32} + c_{12}h_{52})e_6 - h_{12}e_7$, $e_3' = e_3 - c_{32}h_{52}e_6 - h_{32}e_7$, $e_5' = e_5 - h_{52}e_7$. Our algebras assume the form of (128) with $n = 8$ and the b_{ij} replaced by c_{ij} . With these replacements, the discussion on page 50 concerning (128) holds in the present case and we finally have algebra 10 or algebra 11 for $n = 8$.

We have now shown that the algebras 12 and 13 with $n < 7$ lead to no new general algebras under Theorems 7₁ and 8₁ but to the algebras 12 and 13 in n units which had been previously derived from algebras 6 and 7 through (110) and (111) by means of Theorems 7₂ and 8₁.

We observe that algebras (90) and (91) with $n = 5$ are algebras 10 and 11 in 5 units, and that for $n > 5$ they are special cases of algebras 10 and 11 in n units; also that algebras (107) and (108) are special cases of algebras 12 and 13 for $n = 6$, while (110) and (111) are special cases of 12 and 13 for $n > 6$. Hence the only classes of algebras of degree $n - 3 > 1$ which we have thus far discovered are algebras 8, 9, 10, 11, 12, 13. There is, however, one further possibility to be considered. In an algebra of order n arising from one of these six algebras in $n - 1$ units by the use of one of the Theorems 6₂, 7₂, 8₂, it is evident that the basal unit e_r is of degree $n - 4$, but the degree of some other basal unit may be $n - 3$. However, its degree cannot be greater than $n - 3$ in view of the following theorem which is an immediate corollary of Theorem 5.

When a nilpotent algebra in n basal units is derived from a nilpotent algebra N in $n - 1$ units by means of Theorem 5, the degree of any unit e_i of N is increased by at most one.

We are not considering algebras of degree 1 in this chapter, and since e_r has degree at least 2, we have $n - 4 \geq 2$ and hence $n > 5$.

12. Algebras arising from algebras 8 and 9 in $n - 1$ units by the use of Theorem 6₂, but such that the degree of some e_i ($1 < r$) is $n - 3$. The new algebras, if they exist, are of the form (54) with $r = 4$, except that the powers of e are now $e_i^{1-3} = e_i$ ($i = 4, \dots, n - 1$), $e^{n-3} = 0$. Then $e_1 e^2 = 0 = e_1 e_5$, $e_1 e^3 = 0 = e_1 e_6$, ..., $e_1 e^{n-4} = 0 = e_1 e_{n-1}$ ($i = 1, \dots, r - 1$). Hence $e_1^3 = b_{11} e_1 e_{n-1} = 0$, the degree of e_1 is at most 2 and $n - 3 \leq 2$, $n \leq 5$. Therefore the case considered in this section does not arise.

13. Algebras derived from algebras 10 and 11 in $n - 1$ units by means of Theorem 7₂, but such that e_1 or e_2 is of degree $n - 3$. For $n > 6$ these algebras have the form of those in Section 4 except that now we have $e_3^{1-3} = e_1$ ($i = 5, \dots, n - 1$), $e_3^{n-3} = 0$. Then $e_1 e_3^2 = 0 = e_1 e_5$, $e_1 e_3^3 = 0 = e_1 e_6$, ..., $e_1 e_3^{n-4} = 0 = e_1 e_{n-1}$. Hence $e_1^3 = a_{11} e_1 e_4 = a_{11} c_{14} e_{n-1} + a_{11} d_{14} e_n$, $e_1^4 = 0$ and the degree of e_1 is at most 3. But $n - 3 \leq 3$ for $n \leq 6$. Now $e_2 e_3^3 = e_2 e_5 e_3 = 0 = e_2 e_6$, $e_2 e_3^4 = 0 = e_2 e_7$, ..., $e_2 e_3^{n-4} = 0 = e_2 e_{n-1}$. Hence $e_2^3 = a_{22} e_2 e_4 + b_{22} e_2 e_{n-2} = g e_{n-1} + h e_n$ and $e_2^4 = 0$. Thus the degree of e_2 is at most 3, and it is clear that for $n > 6$ the case considered in this section does not arise.

We now consider the case $n = 6$. The algebras in $n - 1$ units from which our new algebras arise are (90) and (91) with $n = 5$. The new algebras have the multiplication table

$$\begin{aligned}
 e_1^2 &= a_{11} e_4 + b_{11} e_5 + c_{11} e_6, & e_2^2 &= a_{22} e_4 + b_{22} e_5 + c_{22} e_6, \\
 e_1 e_2 &= a_{12} e_4 + b_{12} e_5 + c_{12} e_6, & e_2 e_1 &= a_{21} e_4 + b_{21} e_5 + c_{21} e_6, \\
 (153) \quad e_1 e_3 &= c_{13} e_6, & e_3 e_1 &= a_{31} e_4 + b_{31} e_5 + c_{31} e_6, \\
 e_2 e_3 &= a_{23} e_4 + c_{23} e_6, & e_3 e_2 &= a_{32} e_4 + b_{32} e_5 + c_{32} e_6, \\
 e_4 e_1 &= c_{41} e_6, & e_1 e_4 &= c_{14} e_6 & (i = 1, \dots, 5), \\
 e_5 e_1 &= c_{51} e_6, & e_1 e_5 &= c_{15} e_6 & (i = 1, 2), & e_3^2 &= e_5,
 \end{aligned}$$

where e_1 or e_2 is of degree $n - 3 = 3$. The associative law applied to the products $e_4e_3^2$, $e_3^2e_4$, $e_1e_3^2$, $e_1^2e_4$, $e_1e_2e_4$, $e_2e_1e_4$, $e_2^2e_4$, $e_3e_1e_4$, $e_4e_2e_3$, $e_3e_2e_4$, gives

$$(154) \quad e_4e_5 = e_5e_4 = e_1e_5 = 0,$$

$$(155) \quad \begin{aligned} a_{11}e_4^2 &= a_{12}e_4^2 = a_{21}e_4^2 = a_{31}e_4^2 \\ &= a_{23}e_4^2 = a_{32}e_4^2 = a_{22}e_4^2 = 0. \end{aligned}$$

First assume that e_1 is of degree 3. Then $e_1^3 = a_{11}e_1e_4 = a_{11}c_{14}e_6 \neq 0$, and hence $a_{11} \neq 0$, $c_{14} \neq 0$. Therefore by (155), $e_4^2 = 0$. Also $e_1^3 = (a_{11}c_{41} + b_{11}c_{51})e_6$ and $a_{11}c_{14} = a_{11}c_{41} + b_{11}c_{51}$. Now put $e_1' = e_3$, $e_3' = e_1$, $e_4' = e_5$, $e_5' = a_{11}e_4 + b_{11}e_5 + c_{11}e_6$, $e_6' = a_{11}c_{14}e_6$. Under this transformation, the algebras (153) with (154) and $e_4^2 = 0$ take the form

$$\begin{aligned} e_1^2 &= e_4, & e_2^2 &= a_{22}e_4 + b_{22}e_5 + c_{22}e_6, \\ e_1e_2 &= a_{12}e_4 + b_{12}e_5 + c_{12}e_6, & e_2e_1 &= a_{21}e_4 + b_{21}e_5 + c_{21}e_6, \\ e_1e_3 &= a_{13}e_4 + b_{31}e_5 + c_{13}e_6, & e_3e_1 &= c_{31}e_6, \\ e_2e_3 &= a_{23}e_4 + b_{23}e_5 + c_{23}e_6, & e_3e_2 &= a_{32}e_4 + b_{32}e_5 + c_{32}e_6, \\ e_2e_4 &= c_{24}e_6, & e_4e_2 &= c_{42}e_6, & e_4e_3 &= c_{43}e_6, \\ e_1e_5 &= c_{15}e_6, & e_5e_1 &= c_{51}e_6, \\ e_2e_5 &= c_{25}e_6, & e_5e_2 &= c_{52}e_6, & e_3^2 &= e_5, & e_3^3 &= e_6. \end{aligned}$$

From the products $e_3e_1e_3$, $e_3^2e_1$, we find $b_{13} = 0 = c_{51}$. Now put $e_1' = e_1 - c_{31}e_5$, $e_2' = e_2 - b_{32}e_3 - c_{32}e_5$. This transformation eliminates the product e_3e_1 and the terms in e_5 and e_6 in e_3e_2 ; also introduces a term in e_6 into e_1^2 . The products $e_3^2e_2$, $e_3e_1e_2$, $e_3e_2e_1$ give $c_{52} = b_{12} = b_{21} = 0$. From $e_1e_3^2$, $e_3e_2^2$, $e_3e_2e_3$, $e_2e_3^2$ we obtain the relations $c_{15} = a_{13}c_{43}$, $b_{22} = a_{32}c_{42}$, $b_{23} = a_{32}c_{43}$, $c_{25} = c_{43}(a_{23} + a_{32})$. Hence the algebras now assume the form

$$\begin{aligned}
e_1^2 &= e_4 + c_{11}e_6, & e_2^2 &= a_{22}e_4 + a_{32}c_{42}e_5 + c_{22}e_6, \\
e_1e_2 &= a_{12}e_4 + c_{12}e_6, & e_2e_1 &= a_{21}e_4 + c_{21}e_6, \\
e_1e_3 &= a_{13}e_4 + c_{13}e_6, & e_2e_3 &= a_{23}e_4 + a_{32}c_{43}e_5 + c_{23}e_6, \\
e_2e_4 &= c_{24}e_6, & e_4e_2 &= c_{42}e_6, & e_4e_3 &= c_{43}e_6, & e_1e_5 &= a_{13}c_{43}e_6, \\
e_3e_2 &= a_{32}e_4, & e_2e_5 &= c_{43}(a_{23} + a_{32})e_6, & e_3^2 &= e_5, & e_3^3 &= e_6,
\end{aligned}$$

a case of the reciprocal of algebra 10 or of algebra 11 with $n = 6$.

Now suppose that e_2 is of degree 3 in (153) with (154). If $a_{23} = 0$, we can interchange e_1 and e_2 and obtain the case which has been considered, e_1 of degree 3. Hence assume $a_{23} \neq 0$. Then by (155), $e_4^2 = 0$. Now $e_2^3 = a_{22}e_2e_4 + b_{22}e_2e_5 = (a_{22}c_{24} + b_{22}c_{25})e_6 = (a_{22}c_{42} + b_{22}c_{52})e_6 \neq 0$, and hence

$$(156) \quad a_{22}(c_{24} - c_{42}) + b_{22}(c_{25} - c_{52}) = 0.$$

We first assume $a_{22} = 0$. Then $b_{22} \neq 0$, $c_{25} = c_{52} \neq 0$, $e_2^3 = b_{22}c_{25}e_6$. Now put $e_1' = e_1 - b_{22}^{-1}b_{21}e_2 - a_{23}^{-1}a_{21}e_3$, $e_2' = e_3$, $e_3' = e_2$, $e_5' = b_{22}e_5 + c_{22}e_6$, $e_6' = b_{22}c_{25}e_6$. The algebras (153), (154) with $e_4^2 = 0$ become

$$\begin{aligned}
e_1^2 &= a_{11}e_4 + b_{11}e_5 + c_{11}e_6, & e_2^2 &= b_{22}e_5 + c_{22}e_6, \\
e_1e_2 &= a_{12}e_4 + b_{12}e_5 + c_{12}e_6, & e_2e_1 &= a_{21}e_4 + b_{21}e_5 + c_{21}e_6, \\
e_1e_3 &= a_{13}e_4 + b_{13}e_5 + c_{13}e_6, & e_3e_1 &= c_{31}e_6, \\
e_2e_3 &= a_{23}e_4 + b_{23}e_5 + c_{23}e_6, & e_3e_2 &= a_{32}e_4 + c_{32}e_6, \\
e_1e_4 &= c_{14}e_6, & e_4e_1 &= c_{41}e_6 & (i = 1, 2, 3), \\
e_1e_5 &= c_{15}e_6, & e_5e_1 &= c_{51}e_6, & e_3^2 &= e_5, & e_3^3 &= e_6.
\end{aligned}$$

We observe that $e_3^2e_1 = 0 = e_5e_1$. The products e_3e_1 and e_3e_4 are eliminated by the transformation

$$\begin{aligned}
(158) \quad e_1' &= e_1 - c_{31}e_5, & e_4' &= e_4 - c_{34}e_5, \\
e_2' &= e_2 - (a_{32}c_{34} + d)e_3 - c_{32}e_5,
\end{aligned}$$

with $d = 0$, and (157) with $e_5e_1 = 0$ assumes the form

$$\begin{aligned}
 (159) \quad e_1^2 &= a_{11}e_4 + b_{11}e_5 + c_{11}e_6, & e_2^2 &= a_{22}e_4 + b_{22}e_5 + c_{22}e_6, \\
 e_1e_2 &= a_{12}e_4 + b_{12}e_5 + c_{12}e_6, & e_2e_1 &= a_{21}e_4 + b_{21}e_5 + c_{21}e_6, \\
 e_1e_3 &= a_{13}e_4 + b_{13}e_5 + c_{13}e_6, & e_2e_3 &= a_{23}e_4 + b_{23}e_5 + c_{23}e_6, \\
 e_3e_2 &= a_{32}e_4, & e_1e_4 &= c_{14}e_6, & e_4e_1 &= c_{41}e_6, & e_4e_3 &= c_{43}e_6, \\
 e_2e_4 &= c_{24}e_6, & e_4e_2 &= c_{42}e_6, & e_2e_5 &= c_{25}e_6, & e_5e_2 &= c_{52}e_6, \\
 e_1e_5 &= c_{15}e_6, & e_3^2 &= e_5, & e_3^3 &= e_6.
 \end{aligned}$$

The products $e_3^2e_2$, $e_3e_1e_2$, $e_3e_1^2$, $e_3e_1e_3$ give $c_{52} = b_{12} = b_{11} = b_{13} = 0$. From $e_1e_3^2$, $e_3e_2e_1$, $e_3e_2^2$, $e_3e_2e_3$, $e_2e_3^2$ we obtain the relations $c_{15} = a_{13}c_{43}$, $b_{21} = a_{32}c_{41}$, $b_{22} = a_{32}c_{42}$, $b_{23} = a_{32}c_{43}$, $c_{25} = c_{43}(a_{23} + a_{32})$. Hence (159) becomes

$$\begin{aligned}
 e_1^2 &= a_{11}e_4 + c_{11}e_6, & e_2^2 &= a_{22}e_4 + a_{32}c_{42}e_5 + c_{22}e_6, \\
 e_1e_2 &= a_{12}e_4 + c_{12}e_6, & e_2e_1 &= a_{21}e_4 + a_{32}c_{41}e_5 + c_{21}e_6, \\
 e_1e_3 &= a_{13}e_4 + c_{13}e_6, & e_2e_3 &= a_{23}e_4 + a_{32}c_{43}e_5 + c_{23}e_6, \\
 e_3e_2 &= a_{32}e_4, & e_1e_4 &= c_{14}e_6, & e_4e_1 &= c_{41}e_6, & e_4e_3 &= c_{43}e_6, \\
 e_2e_4 &= c_{24}e_6, & e_4e_2 &= c_{42}e_6, & e_1e_5 &= a_{13}c_{43}e_6, \\
 e_2e_5 &= c_{43}(a_{23} + a_{32})e_6, & e_3^2 &= e_5, & e_3^3 &= e_6,
 \end{aligned}$$

which is the reciprocal of algebra 10 or of algebra 11 with $n = 6$.

Next we suppose $a_{22} \neq 0$ in (153) with (154) and $e_4^2 = 0$.

If $b_{22} \neq 0$, we subject the algebras to the transformation

$$\begin{aligned}
 (160) \quad e_1' &= e_1 - ge_2 + he_3, & e_4' &= e_5, \\
 e_2' &= e_3, & e_5' &= a_{22}e_4 + b_{22}e_5 + c_{22}e_6, \\
 e_3' &= e_2, & e_6' &= (a_{22}c_{24} + b_{22}c_{25})e_6,
 \end{aligned}$$

with $g = b_{22}^{-1}b_{21}$, $h = a_{23}^{-1}b_{22}^{-1}b_{21}a_{22} - a_{23}^{-1}a_{21}$ and obtain

$$\begin{aligned}
 (161) \quad & e_1^2 = a_{11}e_4 + b_{11}e_5 + c_{11}e_6, \quad e_2^2 = e_4, \\
 & e_1e_2 = a_{12}e_4 + b_{12}e_5 + c_{12}e_6, \quad e_2e_1 = a_{21}e_4 + b_{21}e_5 + c_{21}e_6, \\
 & e_1e_3 = a_{13}e_4 + b_{13}e_5 + c_{13}e_6, \quad e_3e_1 = c_{31}e_6, \\
 & e_2e_3 = a_{23}e_4 + b_{23}e_5 + c_{23}e_6, \quad e_3e_2 = a_{32}e_4 + b_{32}e_5 + c_{32}e_6, \\
 & e_3^2 = e_5, \quad e_1e_4 = c_{14}e_6, \quad e_4e_1 = c_{41}e_6 \quad (i = 1, 3), \\
 & e_3^3 = e_6, \quad e_1e_5 = c_{15}e_6, \quad e_5e_1 = c_{51}e_6 \quad (i = 1, 2)
 \end{aligned}$$

in which $e_3^2e_1 = 0 = e_5e_1$. We then apply (158) with $d = b_{32}$ and obtain (159) which proves to be the reciprocal of algebra 10 or algebra 11. If $b_{22} = 0$ we see from (156) and $e_2^3 \neq 0$ that $c_{24} = c_{42} \neq 0$. We now use (160) with $g = a_{22}^{-1}a_{21}$, $h = 0$ and obtain algebras of the same form as (161) with $a_{12} = a_{32} = 0$, except that now $e_3e_1 = a_{31}e_4 + c_{31}e_6$. We next put $e_1' = e_1 - a_{31}c_{34}e_3 - c_{31}e_5$, $e_2' = e_2 - b_{32}e_3 - c_{32}e_5$, $e_4' = e_4 - c_{34}e_5$ and find our algebras in the form

$$\begin{aligned}
 & e_1^2 = a_{11}e_4 + b_{11}e_5 + c_{11}e_6, \quad e_2^2 = a_{22}e_4 + b_{22}e_5 + c_{22}e_6, \\
 & e_1e_2 = a_{12}e_4 + b_{12}e_5 + c_{12}e_6, \quad e_2e_1 = a_{21}e_4 + b_{21}e_5 + c_{21}e_6, \\
 & e_1e_3 = a_{13}e_4 + b_{13}e_5 + c_{13}e_6, \quad e_3e_1 = a_{31}e_4, \\
 & e_2e_3 = a_{23}e_4 + b_{23}e_5 + c_{23}e_6, \quad e_4e_3 = c_{43}e_6, \\
 & e_3^2 = e_5, \quad e_1e_4 = c_{14}e_6, \quad e_4e_1 = c_{41}e_6, \\
 & e_3^3 = e_6, \quad e_1e_5 = c_{15}e_6, \quad e_5e_1 = c_{51}e_6, \quad (i = 1, 2).
 \end{aligned}$$

We note that $e_3^2e_1 = 0 = e_5e_1$, $e_3^2e_2 = 0 = e_5e_2$. If we now interchange e_1 and e_2 the algebras assume the form (159) with $e_5e_2 = 0$ and we therefore have the reciprocal of algebra 10 or of algebra 11.

We have now shown that an algebra derived from algebra 10 or algebra 11 in 5 units by means of Theorem 7₂, but such that e_1 or e_2 is of degree 3, is equivalent to the reciprocal of algebra 10 or of algebra 11 in 6 units.

14. Algebras arising from algebras 12 and 13 in $n - 1$ units by the use of Theorem 7₂ or 8₂ but such that e_1 is of degree $n - 3$. We first consider the case $n = 6$. The algebras arise from 12 and 13 in 5 units by the use of Theorem 7₂ and have the multiplication table

$$\begin{aligned}
 (162) \quad & e_1^2 = a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6, \\
 & e_1e_2 = a_{12}e_3 + c_{12}e_5 + d_{12}e_6, \\
 & e_2e_1 = a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6, \\
 & e_1e_3 = c_{13}e_5 + d_{13}e_6, \quad e_3e_1 = c_{31}e_5 + d_{31}e_6, \\
 & e_2e_3 = c_{23}e_5 + d_{23}e_6, \quad e_3e_2 = c_{32}e_5 + d_{32}e_6, \\
 & e_1e_4 = c_{14}e_5 + d_{14}e_6, \quad e_4e_1 = c_{41}e_5 + d_{41}e_6, \\
 & e_3e_4 = d_{34}e_6, \quad e_4e_3 = d_{43}e_6, \quad e_3^2 = d_{33}e_6, \quad e_2^2 = e_4, \\
 & e_1e_5 = d_{15}e_6, \quad e_5e_1 = d_{51}e_6 \quad (i = 1, 2, 3, 5)
 \end{aligned}$$

in which e_1 is of degree $n - 3 = 3$. The products e_4e_5 , e_5e_4 are omitted because $e_2^2e_5 = 0 = e_4e_5$, $e_5e_2^2 = 0 = e_5e_4$. Now $e_1^3 = (a_{11}c_{13} + b_{11}c_{14})e_5 + (a_{11}d_{13} + b_{11}d_{14} + c_{11}d_{15})e_6$. But algebras 12 and 13 in 5 units are of degree 2, and in each of them $e_1^3 = (a_{11}c_{13} + b_{11}c_{14})e_5 = 0 = (a_{11}d_{13} + b_{11}d_{14} + c_{11}d_{15})e_6$. Hence in the present case $e_1^3 = (a_{11}d_{13} + b_{11}d_{14} + c_{11}d_{15})e_6 = (a_{11}d_{31} + b_{11}d_{41} + c_{11}d_{51})e_6 \neq 0$. Evidently a_{11} , b_{11} , c_{11} cannot all be zero.

If $a_{11} \neq 0$, put $e_1' = e_2$, $e_2' = e_1$, $e_3' = e_4$, $e_4' = a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6$, $e_6' = (a_{11}d_{13} + b_{11}d_{14} + c_{11}d_{15})e_6$. The algebras (162) become

$$\begin{aligned}
 (163) \quad & e_1^2 = e_3, & e_1e_2 &= a_{12}e_3 + b_{12}e_4 + c_{12}e_5 + d_{12}e_6, \\
 & e_2e_1 = a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6, \\
 & e_2e_3 = c_{23}e_5 + d_{23}e_6, & e_3e_2 &= c_{32}e_5 + d_{32}e_6, \\
 & e_1e_4 = c_{14}e_5 + d_{14}e_6, & e_4e_1 &= c_{41}e_5 + d_{41}e_6, \\
 & e_3e_4 = d_{34}e_6, & e_4e_3 &= d_{43}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6, \\
 & e_1e_5 = d_{15}e_6, & e_5e_1 &= d_{51}e_6, & & & (i = 1, 2, 5),
 \end{aligned}$$

the products e_4e_5 and e_5e_4 being omitted because $e_2^2e_5 = 0 = e_4e_5$, $e_5e_2^2 = 0 = e_5e_4$. Now under the transformation

$$(164) \quad e_1' = e_1 - (a_{12}d_{32} + b_{12})e_2 - d_{12}e_4, \quad e_3' = e_3 - d_{32}e_4$$

the algebras (163) take the form (131) with $e_5e_3 = 0 = e_3e_5$.

From (131) we obtain algebras 12 and 13 in 6 units, or for certain values of the coefficients, algebras 8, 9, 10, or 11.

Now assume $a_{11} = 0$, $b_{11} \neq 0$ in (162). Put $e_1' = e_2$, $e_2' = e_1$, $e_4' = b_{11}e_4 + c_{11}e_5 + d_{11}e_6$, $e_6' = (b_{11}d_{14} + c_{11}d_{15})e_6$. The algebras (162) assume the form

$$\begin{aligned}
 & e_1^2 = b_{11}e_4 + c_{11}e_5 + d_{11}e_6, \\
 & e_1e_2 = a_{12}e_3 + b_{12}e_4 + c_{12}e_5 + d_{12}e_6, \\
 & e_2e_1 = a_{21}e_3 + c_{21}e_5 + d_{21}e_6, \\
 & e_1e_3 = c_{13}e_5 + d_{13}e_6, & e_3e_1 &= c_{31}e_5 + d_{31}e_6, \\
 & e_2e_3 = c_{23}e_5 + d_{23}e_6, & e_3e_2 &= c_{32}e_5 + d_{32}e_6, \\
 & e_1e_4 = d_{14}e_6, & e_4e_1 &= d_{41}e_6, & e_3e_4 &= d_{34}e_6, & e_4e_3 &= d_{43}e_6, \\
 & e_1e_5 = d_{15}e_6, & e_5e_1 &= d_{51}e_6 & & (i = 1, 2, 3, 5), \\
 & e_3^2 = d_{33}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6,
 \end{aligned}$$

where e_4e_5 , e_5e_4 are omitted because $e_2^2e_5 = 0 = e_4e_5$, $e_5e_2^2 = 0 = e_5e_4$. Applying (164) to these algebras we again obtain (131) with $c_{14} = 0 = c_{41}$.

Finally, we assume $a_{11} = b_{11} = 0$, $c_{11} \neq 0$ in (162). Set $e_1' = e_2$, $e_2' = e_1$, $e_4' = c_{11}e_5 + d_{11}e_6$, $e_5' = e_4$, $e_6' = c_{11}d_{15}e_6$. Under this transformation, (162) becomes

$$\begin{aligned}
 (165) \quad & e_1^2 = e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + d_{21}e_6, \\
 & e_1e_2 = a_{12}e_3 + b_{12}e_4 + c_{12}e_5 + d_{12}e_6, \\
 & e_1e_3 = b_{13}e_4 + d_{13}e_6, & e_3e_1 &= b_{31}e_4 + d_{31}e_6, \\
 & e_2e_3 = b_{23}e_4 + d_{23}e_6, & e_3e_2 &= b_{32}e_4 + d_{32}e_6, \\
 & e_1e_4 = d_{14}e_6, & e_4e_1 &= d_{41}e_6, & e_2e_5 &= b_{25}e_4 + d_{25}e_6, \\
 & e_2^2 = e_4, & e_2^3 &= e_6, & e_5e_2 &= b_{52}e_4 + d_{52}e_6, \\
 & e_1e_3 = d_{13}e_6, & e_3e_1 &= d_{31}e_6 & (i = 3, 4, 5).
 \end{aligned}$$

Since $0 = e_4e_5 = e_2^2e_5 = b_{25}e_2e_4 = b_{25}e_6$, we have $b_{25} = 0$, and similarly, $b_{52} = 0$. Now put $e_1' = e_1 - (b_{12} + a_{12}d_{32} + c_{12}d_{52})e_2 - d_{12}e_4$, $e_3' = e_3 - d_{32}e_4$, $e_5' = -d_{52}e_4 + e_5$. This transformation eliminates b_{12} , d_{12} , d_{32} , d_{52} and introduces a few terms so that (165) assumes the form

$$\begin{aligned}
 (166) \quad & e_1^2 = a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6, \\
 & e_1e_2 = a_{12}e_3 + c_{12}e_5, & e_2e_1 &= a_{21}e_3 + b_{21}e_4 + d_{21}e_6, \\
 & e_1e_3 = b_{13}e_4 + d_{13}e_6, & e_3e_1 &= b_{31}e_4 + d_{31}e_6, \\
 & e_2e_3 = b_{23}e_4 + d_{23}e_6, & e_3e_2 &= b_{32}e_4, \\
 & e_1e_4 = d_{14}e_6, & e_4e_1 &= d_{41}e_6, & e_1e_5 &= d_{15}e_6, & e_5e_1 &= d_{51}e_6, \\
 & e_2e_5 = d_{25}e_6, & e_2^2 &= e_4, & e_2^3 &= e_6, \\
 & e_3e_1 = d_{31}e_6, & e_1e_3 &= d_{13}e_6 & (i = 3, 4, 5).
 \end{aligned}$$

From $e_1e_2^2$ we find $d_{14} = 0$. From $e_3e_2^2$ and $e_3^2e_2$ we have $b_{32} = d_{34} = 0$. The products $e_2e_3e_2$, $e_2e_3e_1$, $e_1e_3e_2$, and $e_2^2e_3$ give $b_{23} = b_{31} = b_{13} = d_{43} = 0$. The algebras are now seen to be a case of algebras (131).

We next consider the case $n = 7$. The new algebras are derived from 12 and 13 in 6 units by means of Theorem 8₂ and have

the form (145) except that $e_2^4 = 0$. Using the associative law we have $e_2^2 e_5 = 0 = e_4 e_5$, $e_5 e_2^2 = 0 = e_5 e_4$, $e_2^3 e_5 = 0 = e_6 e_5$, $e_5 e_2^3 = 0 = e_5 e_6$, $e_3 e_4 e_2 = 0 = e_3 e_6$, $e_2 e_4 e_3 = 0 = e_6 e_3$. Hence the algebras take the form

$$\begin{aligned}
 e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6 + g_{11}e_7, \\
 e_1e_2 &= a_{12}e_3 + c_{12}e_5 + g_{12}e_7, \\
 e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7, \\
 e_1e_3 &= c_{13}e_5 + d_{13}e_6 + g_{13}e_7, \quad e_3e_1 = c_{31}e_5 + d_{31}e_6 + g_{31}e_7, \\
 e_2e_3 &= c_{23}e_5 + d_{23}e_6 + g_{23}e_7, \quad e_3e_2 = c_{32}e_5 + g_{32}e_7, \\
 (167) \quad e_1e_4 &= c_{14}e_5 + g_{14}e_7, \quad e_4e_1 = c_{41}e_5 + d_{41}e_6 + g_{41}e_7, \\
 e_1e_5 &= d_{15}e_6 + g_{15}e_7, \quad e_5e_1 = d_{51}e_6 + g_{51}e_7, \\
 e_2e_5 &= d_{25}e_6 + g_{25}e_7, \quad e_5e_2 = g_{52}e_7, \\
 e_3e_4 &= g_{34}e_7, \quad e_4e_3 = d_{43}e_6 + g_{43}e_7, \\
 e_1e_6 &= g_{16}e_7, \quad e_6e_1 = g_{61}e_7, \quad e_3e_5 = g_{35}e_7, \quad e_5e_3 = g_{53}e_7, \\
 e_3^2 &= d_{33}e_6 + g_{33}e_7, \quad e_5^2 = g_{55}e_7, \quad e_2^2 = e_4, \quad e_2^3 = e_6.
 \end{aligned}$$

From the product e_1^3 in (167) we have $e_1^3 = (a_{11}c_{13} + b_{11}c_{14})e_5 + k_6e_6 + k_7e_7$, where

$$\begin{aligned}
 k_6 &= a_{11}d_{13} + c_{11}d_{15} = a_{11}d_{31} + b_{11}d_{41} + c_{11}d_{51}, \\
 (168) \quad k_7 &= a_{11}g_{13} + b_{11}g_{14} + c_{11}g_{15} + d_{11}g_{16} \\
 &= a_{11}g_{31} + b_{11}g_{41} + c_{11}g_{51} + d_{11}g_{61}.
 \end{aligned}$$

Since algebras 12 and 13 in n units are always of degree $n - 3$, we have for $n = 5$, $e_1^3 = 0$; then for $n = 6$, $e_1^3 = he_6$, $e_1^4 = 0$. Hence in the above expression for e_1^3 the coefficient of e_5 must be zero and $e_1^4 = k_6e_1e_6 = (a_{11}d_{13} + c_{11}d_{15})g_{16}e_7 = (a_{11}d_{13} + c_{11}d_{15})g_{61}e_7 \neq 0$. Evidently a_{11} and c_{11} cannot both be zero. Also we must have $g_{16} = g_{61} \neq 0$.

If $a_{11} \neq 0$, subject (167) to the transformation $e_1' = e_2$, $e_2' = e_1$, $e_3' = e_4$, $e_4' = a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6 + g_{11}e_7$, $e_6' = k_6e_6 + k_7e_7$, $e_7' = k_6g_{16}e_7$. The algebras become

$$\begin{aligned}
 (169) \quad & e_1^2 = e_3, \\
 & e_1e_2 = a_{12}e_3 + b_{12}e_4 + c_{12}e_5 + d_{12}e_6 + g_{12}e_7, \\
 & e_2e_1 = a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7, \\
 & e_1e_3 = d_{13}e_6 + g_{13}e_7 = e_3e_1, \\
 & e_2e_3 = c_{23}e_5 + g_{23}e_7, \quad e_3e_2 = c_{32}e_5 + d_{32}e_6 + g_{32}e_7, \\
 & e_1e_4 = c_{14}e_5 + d_{14}e_6 + g_{14}e_7, \quad e_4e_1 = c_{41}e_5 + d_{41}e_6 + g_{41}e_7, \\
 & e_1e_5 = d_{15}e_6 + g_{15}e_7, \quad e_5e_1 = g_{51}e_7, \\
 & e_2e_5 = d_{25}e_6 + g_{25}e_7, \quad e_5e_2 = d_{52}e_6 + g_{52}e_7, \\
 & e_4e_3 = g_{43}e_7, \quad e_4e_5 = g_{45}e_7, \quad e_5e_4 = g_{54}e_7, \quad e_5^2 = g_{55}e_7, \\
 & e_3e_4 = d_{34}e_6 + g_{34}e_7, \quad e_2^2 = e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7.
 \end{aligned}$$

From the products $e_3e_4e_2$, $e_1e_5e_2$ we have $d_{34} = 0 = d_{15}$. From $e_2e_5e_2$ we find

$$(170) \quad d_{25} = d_{52}.$$

The products $e_3e_2^2$, $e_2e_3e_2$, $e_1e_4e_2$, $e_2e_4e_1$ with (170) give the relations

$$(171) \quad c_{23}d_{25} = c_{32}d_{25} = c_{14}d_{25} = c_{41}d_{25} = 0.$$

Now suppose that $d_{25} \neq 0$ in (169) with $d_{15} = 0 = d_{34}$. Then by (171) $c_{23} = c_{32} = c_{14} = c_{41} = 0$. From $e_2^2e_5$ and $e_5e_2^2$ with (170) we obtain

$$(172) \quad g_{45} = g_{54} = d_{25} = d_{52}.$$

The products $e_1e_3e_2$, $e_2e_3e_2$, $e_1e_4e_2$, $e_2e_4e_1$ give $d_{13} = d_{32} = d_{14} = d_{41} = 0$. From $e_1e_2^2$ we have

$$(173) \quad b_{12} + c_{12}d_{52} = 0, \quad g_{14} = d_{12} + a_{12}g_{32} + c_{12}g_{52}.$$

Now put $e_1' = e_1 - (d_{12} + a_{12}g_{32} + c_{12}g_{52})e_4 - g_{12}e_6$, $e_3' = e_3 - g_{32}e_6$, $e_5' = -d_{52}e_4 + e_5 - g_{52}e_6$, keeping (172) and (173) in mind. The algebras now take the form

$$\begin{aligned} e_1^2 &= a_{11}e_3 + d_{11}e_6 + g_{11}e_7, & e_1e_2 &= a_{12}e_3 + c_{12}e_5, \\ e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7, \\ e_1e_3 &= g_{13}e_7, & e_3e_1 &= g_{31}e_7, & e_2e_3 &= g_{23}e_7, & e_4e_1 &= g_{41}e_7, \\ e_1e_5 &= g_{15}e_7, & e_5e_1 &= g_{51}e_7, & e_3e_4 &= g_{34}e_7, & e_4e_3 &= g_{43}e_7, \\ e_3e_5 &= g_{35}e_7, & e_5e_3 &= g_{53}e_7, & e_2e_5 &= g_{25}e_7, \\ e_5^2 &= g_{55}e_7, & e_2^2 &= e_4, & e_2^3 &= e_6, & e_2^4 &= e_7. \end{aligned}$$

By the associative law $e_2^2e_3 = 0 = e_4e_3$, $e_3e_2^2 = 0 = e_3e_4$. From the products $e_1^2e_2$, $e_2e_1e_2$, $e_2^2e_1$ we have $d_{11} = a_{12}g_{13} + c_{12}g_{15}$, $b_{21} = 0$, $d_{21} = a_{12}g_{23} + c_{12}g_{25}$, $g_{41} = g_{23}(a_{12} + a_{21}) + g_{25}(c_{12} + c_{21})$. The algebras are now seen to be in the form (148) with $c_{11} = c_{13} = c_{31} = c_{23} = c_{32} = d_{33} = g_{33} = 0$. From (148) we obtain algebras 12 and 13, or in some cases, algebras 8 and 9 or 10 and 11.

Next, assume $d_{25} = 0$ in (169) with $d_{15} = d_{34} = 0$. Then $d_{52} = 0$, $e_2^2e_5 = 0 = e_4e_5$, $e_5e_2^2 = 0 = e_5e_4$. Now put

$$\begin{aligned} (174) \quad e_1' &= e_1 - (b_{12} + a_{12}d_{32} + a_{12}c_{32}g_{52})e_2 \\ &\quad - (d_{12} + a_{12}g_{32} + c_{12}g_{52})e_4 - g_{12}e_6, \\ e_3' &= e_3 - (d_{32} + c_{32}g_{52})e_4 - g_{32}e_6, \\ e_5' &= e_5 - g_{52}e_6. \end{aligned}$$

The algebras now assume the form

$$\begin{aligned} (175) \quad e_1^2 &= a_{11}e_3 + b_{11}e_4 + c_{11}e_5 + d_{11}e_6 + g_{11}e_7, \\ e_1e_2 &= a_{12}e_3 + c_{12}e_5, \\ e_2e_1 &= a_{21}e_3 + b_{21}e_4 + c_{21}e_5 + d_{21}e_6 + g_{21}e_7, \\ e_1e_3 &= c_{13}e_5 + d_{13}e_6 + g_{13}e_7, & e_3e_1 &= c_{31}e_5 + d_{31}e_6 + g_{31}e_7, \\ e_2e_3 &= c_{23}e_5 + d_{23}e_6 + g_{23}e_7, & e_3e_2 &= c_{32}e_5, \end{aligned}$$

$$\begin{aligned}
 (175) \quad & e_1 e_4 = c_{14} e_5 + d_{14} e_6 + g_{14} e_7, \quad e_4 e_1 = c_{41} e_5 + d_{41} e_6 + g_{41} e_7, \\
 & e_1 e_5 = g_{15} e_7, \quad e_5 e_1 = g_{51} e_7, \quad e_3 e_4 = g_{34} e_7, \quad e_4 e_3 = g_{43} e_7, \\
 & e_1 e_6 = g_{16} e_7, \quad e_6 e_1 = g_{61} e_7, \quad e_2 e_5 = g_{25} e_7, \quad e_3^2 = g_{33} e_7, \\
 & e_5^2 = g_{55} e_7, \quad e_2^2 = e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7.
 \end{aligned}$$

From $e_1 e_2^2$ we find $d_{14} = g_{14} = 0$, $c_{14} = a_{12} c_{32}$. All of the relations on page 57 after (146) obtained by means of the associative law hold in the present case and we have (148) with $g_{35} = g_{53} = d_{33} = 0$.

Now we assume $a_{11} = 0$ in (167). Then $c_{11} \neq 0$ and $e_1^4 = c_{11} d_{15} g_{16} e_7 \neq 0$. The transformation $e_1' = e_2$, $e_2' = e_1$, $e_4' = b_{11} e_4 + c_{11} e_5 + d_{11} e_6 + g_{11} e_7$, $e_5' = e_4$, $e_6' = k_6 e_6 + k_7 e_7$, $e_7' = c_{11} d_{15} g_{16} e_7$, with k_6 and k_7 defined by (168), gives (167) the form

$$\begin{aligned}
 (176) \quad & e_1^2 = e_5, \quad e_1 e_2 = a_{12} e_3 + b_{12} e_4 + c_{12} e_5 + d_{12} e_6 + g_{12} e_7, \\
 & e_2 e_1 = a_{21} e_3 + b_{21} e_4 + c_{21} e_5 + d_{21} e_6 + g_{21} e_7, \\
 & e_1 e_3 = b_{13} e_4 + c_{13} e_5 + d_{13} e_6 + g_{13} e_7, \\
 & e_3 e_1 = b_{31} e_4 + c_{31} e_5 + d_{31} e_6 + g_{31} e_7, \\
 & e_2 e_3 = b_{23} e_4 + c_{23} e_5 + d_{23} e_6 + g_{23} e_7, \\
 & e_3 e_2 = b_{32} e_4 + c_{32} e_5 + d_{32} e_6 + g_{32} e_7, \\
 & e_1 e_4 = d_{14} e_6 + g_{14} e_7, \quad e_4 e_1 = d_{41} e_6 + g_{41} e_7, \\
 & e_1 e_5 = d_{15} e_6 + g_{15} e_7 = e_5 e_1, \\
 & e_2 e_5 = b_{25} e_4 + c_{25} e_5 + d_{25} e_6 + g_{25} e_7, \\
 & e_5 e_2 = b_{52} e_4 + c_{52} e_5 + d_{52} e_6 + g_{52} e_7, \\
 & e_3 e_4 = g_{34} e_7, \quad e_4 e_3 = d_{43} e_6 + g_{43} e_7, \\
 & e_3 e_5 = g_{35} e_7, \quad e_5 e_3 = d_{53} e_6 + g_{53} e_7, \\
 & e_3^2 = d_{33} e_6 + g_{33} e_7, \quad e_2^2 = e_4, \quad e_2^3 = e_6, \quad e_2^4 = e_7.
 \end{aligned}$$

By the associative law $0 = e_4 e_5 = e_2^2 e_5 = b_{25} e_2 e_4 + c_{25} e_2 e_5 + d_{25} e_2 e_6 = b_{25} e_6 + c_{25} (b_{25} e_4 + c_{25} e_5 + d_{25} e_6 + g_{25} e_7) + d_{25} e_7 =$

$b_{25}c_{25}e_4 + c_{25}^2e_5 + (b_{25} + c_{25}d_{25})e_6 + (d_{25} + c_{25}g_{25})e_7$. Hence $c_{25} = 0 = b_{25} = d_{25}$. Similarly $c_{52} = 0 = b_{52} = d_{52}$. Using the products $e_2e_5e_3, e_2e_4e_3, e_1e_4e_2, e_2e_4e_1, e_1e_3e_4, e_4e_3e_1, e_2e_3e_4, e_4e_3e_2, e_1e_5e_2$, we find $d_{53} = d_{43} = d_{14} = d_{41} = b_{13} = b_{31} = b_{23} = b_{32} = d_{15} = 0$. We now use the transformation (174) which eliminates from (176) the product e_5e_2 , the terms in e_4 and e_6 in e_1e_2 and the terms in e_6 and e_7 in e_3e_2 , but introduces a few terms. The algebras are now in the form of (175) with $c_{14} = c_{41} = g_{55} = 0, g_{16} = g_{61}$, with the added products $e_3e_5 = g_{35}e_7, e_5e_3 = g_{53}e_7$, and with $e_3^2 = d_{33}e_6 + g_{33}e_7$. As in (175) we have $d_{14} = g_{14} = 0$. The relations on page 57 after (146) obtained by means of the associative law hold here with $c_{14} = c_{41} = g_{61} = 0$. Hence we again have a case of (148).

We now consider the case $n = 8$. The new algebras are derived from 12 and 13 in 7 units by means of Theorem 8₂ and are of the form (152) except that $e_2^5 = 0$. By the associative law $e_1e_6e_2 = 0 = e_1e_7$. Then $e_1^3 = a_{11}e_1e_3 + b_{11}e_1e_4 + c_{11}e_1e_5 + d_{11}e_1e_6 = (a_{11}c_{13} + b_{11}c_{14})e_5 + a_{11}d_{13}e_6 + (a_{11}g_{13} + c_{11}g_{15})e_7 + (a_{11}h_{13} + b_{11}h_{14} + c_{11}h_{15} + d_{11}h_{16})e_8$. Then $e_1^4 = (a_{11}c_{13} + b_{11}c_{14})e_1e_5 + a_{11}d_{13}e_1e_6 = ce_7 + de_8, e_1^5 = 0$ and the degree of e_1 is at most 4. But for $n = 8, n - 3 = 5$ and hence e_1 is not of degree $n - 3$.

In the case $n > 8$ the new algebras are of the form (127) except that the powers of e_2 are now $e_2^2 = e_4, e_2^{1-3} = e_1$ ($i = 6, \dots, n - 1$), $e_2^{n-3} = 0$. By the associative law $e_1e_6e_2 = 0 = e_1e_2^4 = e_1e_7, e_1e_2^5 = 0 = e_1e_8, \dots, e_1e_2^{n-4} = 0 = e_1e_{n-1}$. Then $e_1^3 = a_{11}e_1e_3 + b_{11}e_1e_5 + c_{11}e_1e_{n-3} = a_{11}b_{13}e_5 + a_{11}d_{13}e_{n-2} + (a_{11}g_{13} + b_{11}g_{15})e_{n-1} + (a_{11}h_{13} + b_{11}h_{15} + c_{11}h_{1, n-3})e_n, e_1^4 = a_{11}b_{13}e_1e_5 = ce_{n-1} + de_n, e_1^5 = 0$ and the degree of e_1 is at most 4 and cannot be $n - 3$.

We have now shown that any algebra arising from an algebra 12 or 13 in $n - 1$ units by the use of Theorem 7₂ or 8₂ but such that e_1 is of degree $n - 3$ is equivalent to an algebra 12 or 13 in n units, or in some cases to one of the algebras 8, 9, 10, 11 in n units.

The results of this chapter show that all nilpotent algebras of degree $n - 3$ are included in the classes of algebras 8, 9, 10, 11, 12 and their reciprocals.

CHAPTER VI

CERTAIN TYPES OF ALGEBRAS OF DEGREE ONE

1. Algebras of type A. We shall refer to a nilpotent algebra of degree 1 as an algebra of type A if its multiplication table has the form

$$(177) \quad \begin{aligned} e_i e_j &= c_{ij} e_n = -e_j e_i \\ (i &= 1, \dots, n-2; j = 2, \dots, n-1; i < j). \end{aligned}$$

In any nilpotent algebra of degree 1 we have $(e_i + e_j)^2 = 0$ and hence $e_i e_j = -e_j e_i$ for every i and j . Evidently an algebra of degree 1 derived from the zero algebra in $n-1$ units by means of Theorem 5 has the form (177).

If all c_{1j} in (177) are zero, the algebra is reducible. Hence we assume that some $c_{1j} \neq 0$ and we may, without loss of generality, suppose that $c_{12} \neq 0$. The transformation

$$\begin{aligned} e_1' &= c_{12}^{-1}(c_{21}e_1 - c_{11}e_2) + e_1 & (i = 3, \dots, n-1), \\ e_n' &= c_{12}e_n \end{aligned}$$

eliminates all products $e_i e_j$ involving e_1 or e_2 except $e_1 e_2 = -e_2 e_1$, and (177) becomes

$$(178) \quad \begin{aligned} e_1 e_2 &= e_n = -e_2 e_1, \\ e_i e_j &= c_{ij} e_n = -e_j e_i \\ (i &= 3, \dots, n-2; j = 4, \dots, n-1; i < j). \end{aligned}$$

If now all c_{3j} in (178) are zero we have a reducible algebra. We may without loss of generality assume $c_{34} \neq 0$ and put

$$\begin{aligned} e_1' &= c_{34}e_1, & e_n' &= c_{34}e_n, \\ e_1' &= c_{34}^{-1}(c_{41}e_3 - c_{31}e_4) + e_1 & (i = 5, \dots, n-1). \end{aligned}$$

The products e_1e_j involving e_3 or e_4 are now zero except $e_3e_4 = -e_4e_3$ and (178) takes the form

$$\begin{aligned} (179) \quad e_1e_2 &= e_n = -e_2e_1, & e_3e_4 &= e_n = -e_4e_3, \\ e_1e_j &= c_{1j}e_n = -e_je_1 \\ & (i = 5, \dots, n-2; j = 6, \dots, n-1; i < j). \end{aligned}$$

Again (179) is reducible if all c_{5j} are zero. Hence assume $c_{56} \neq 0$ and put

$$\begin{aligned} e_1' &= c_{56}e_1 & (i = 1, 3, n), \\ e_1' &= c_{56}^{-1}(c_{61}e_5 - c_{51}e_6) + e_1 & (i = 7, \dots, n-1). \end{aligned}$$

The algebra (179) is now in the form

$$\begin{aligned} e_1e_2 &= e_n = -e_2e_1, & e_3e_4 &= e_n = -e_4e_3, & e_5e_6 &= e_n = -e_6e_5, \\ e_1e_j &= c_{1j}e_n = -e_je_1 & (i = 7, \dots, n-2; j = 8, \dots, n-1; i < j). \end{aligned}$$

Continuing this process we obtain the algebra

$$\begin{aligned} (180) \quad e_1e_2 &= e_n = -e_2e_1, & e_3e_4 &= e_n = -e_4e_3, & e_5e_6 &= e_n = -e_6e_5, \\ & \dots, & e_{n-5}e_{n-4} &= e_n = -e_{n-4}e_{n-5}, \\ e_{n-3}e_{n-2} &= c_{n-3,n-2}e_n = -e_{n-2}e_{n-3}, \\ e_{n-3}e_{n-1} &= c_{n-3,n-1}e_n = -e_{n-1}e_{n-3}, \\ e_{n-2}e_{n-1} &= c_{n-2,n-1}e_n = -e_{n-1}e_{n-2}, \end{aligned}$$

if n is even, or the algebra

$$\begin{aligned} (181) \quad e_1e_2 &= e_n = -e_2e_1, & e_3e_4 &= e_n = -e_4e_3, & \dots, \\ e_{n-4}e_{n-3} &= e_n = -e_{n-3}e_{n-4}, \\ e_{n-2}e_{n-1} &= c_{n-2,n-1}e_n = -e_{n-1}e_{n-2} \end{aligned}$$

in case n is odd.

First suppose n is even. If in (180) $c_{n-3,n-2} = 0 = c_{n-3,n-1}$ the algebra is reducible. Hence we may assume without loss of generality that $c_{n-3,n-2} \neq 0$ and put

$$\begin{aligned} e_1' &= c_{n-3,n-2} e_1 & (i = 1, 3, 5, \dots, n-5, n), \\ e_{n-1}' &= c_{n-3,n-2}^{-1} (c_{n-2,n-1} e_{n-3} - c_{n-3,n-1} e_{n-2}) + e_{n-1}. \end{aligned}$$

The algebra (180) becomes

$$\begin{aligned} e_1 e_2 &= e_n = -e_2 e_1, & e_3 e_4 &= e_n = -e_4 e_3, & \dots, \\ e_{n-5} e_{n-4} &= e_n = -e_{n-4} e_{n-5}, \\ e_{n-3} e_{n-2} &= e_n = -e_{n-2} e_{n-3}, \end{aligned}$$

which is reducible.

Next let n be odd. If in (181) $c_{n-2,n-1} = 0$, the algebra is reducible. Hence assume $c_{n-2,n-1} \neq 0$ and put $e_{n-2}' = c_{n-2,n-1}^{-1} e_{n-2}$. The algebra (181) takes the form

$$(182) \quad e_i e_{i+1} = e_n = -e_{i+1} e_i \quad (i = 1, 3, 5, 7, \dots, n-2; n \text{ odd}),$$

and we have proved

THEOREM 9. A nilpotent algebra of type A in n basal units is reducible if n is even. If n is odd, the multiplication table of the algebra may be written in the form (182).

An algebra N with multiplication table (182) is irreducible. For if it is reducible, it is a direct sum $N_1 \oplus N_2$. Let f_1 be a new set of basal units of N_1 and let g_1 be a new set for N_2 . Then $f_1 = \sum_{j=1}^n c_{1j} e_j$ and $g_1 = \sum_{j=1}^n d_{1j} e_j$. Also $f_1 f_k = \sum_j \sum_s c_{1j} c_{ks} e_j e_s = m e_n$ and similarly $g_1 g_k = r e_n$. Now m and r cannot both be different from zero, since in that case e_n would be common to N_1 and N_2 . Suppose that $r = 0$. Then $g_1 g_k = 0$ and N_2 is a zero algebra. Then m cannot be zero for every product $f_1 f_k$ and hence e_n is in N_1 . We may assume N_1 irreducible, since if it is reducible $N_1 = N_1' \oplus N_2'$ where N_2' is a zero algebra, and

$N = N_1' \oplus N''$, $N'' = N_2' \oplus N_2$, a zero algebra, etc. Now $e_p = \sum_k a_{pk} f_k + \sum_k b_{pk} g_k$ and hence $e_p g_1 = \sum_k a_{pk} f_k g_1 + \sum_k b_{pk} g_k g_1 = 0$. But $e_p g_1 = \sum_j d_{1j} e_p e_j = d_{1,p+1} e_p e_{p+1} = d_{1,p+1} e_n$ if p is odd. In case p is even, $e_p g_1 = d_{1,p-1} e_p e_{p-1} = -d_{1,p-1} e_n$. Hence $d_{1j} = 0$ ($j = 1, \dots, n-1$) and $g_1 = d_{1n} e_n$. But e_n is in N_1 and therefore $d_{1n} = 0$. Hence $g_1 = 0$, N_2 is the zero set and N is irreducible.

2. Algebras of type B. We shall refer to a nilpotent algebra of degree 1 as an algebra of type B if its multiplication table is of the form

$$(183) \quad \begin{array}{lll} e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\ e_1 e_3 = e_{r+2} & & = -e_3 e_1, \\ \dots & \dots & \dots, \\ e_1 e_r = e_{2r-1} & & = -e_r e_1, \\ \dots & \dots & \dots, \\ e_p e_r = e_n & & = -e_r e_p, \end{array}$$

where the subscript of each e_i in the left members of the equations is equal to or less than r . Then $n - r$ is the number of rows in the multiplication table.

We have seen in Chapter II that the only algebra of order 4 arising from algebra 4 in 3 units by the use of Theorem 5 is the reducible algebra $e_1 e_2 = e_3 = -e_2 e_1$. Write this algebra as $e_1 e_2 = e_4 = -e_2 e_1$ and consider the algebras of order 5, degree 1, derived from it by means of Theorem 5. The new algebras take the form

$$\begin{array}{ll} e_1 e_2 = e_4 + c_{12} e_5 = -e_2 e_1, & e_1 e_3 = c_{13} e_5 = -e_3 e_1, \\ e_1 e_4 = c_{14} e_5 = -e_4 e_1, & e_2 e_3 = c_{23} e_5 = -e_3 e_2, \\ e_2 e_4 = c_{24} e_5 = -e_4 e_2, & e_3 e_4 = c_{34} e_5 = -e_4 e_3. \end{array}$$

By the associative law $e_1^2 e_2 = 0 = e_1 e_4$, $e_1 e_2^2 = 0 = e_4 e_2$, $e_1 e_2 e_3 = 0 = e_4 e_3$. If we put $e_4' = e_4 + c_{12} e_5$ the algebras become

$$(184) \quad \begin{aligned} e_1 e_2 = e_4 = -e_2 e_1, & \quad e_1 e_3 = c_{13} e_5 = -e_3 e_1, \\ e_2 e_3 = c_{23} e_5 = -e_3 e_2. \end{aligned}$$

If $c_{13} = c_{23} = 0$ we have a reducible algebra. Hence assume $c_{13} \neq 0$ and put $e_2' = -c_{13}^{-1} c_{23} e_1 + e_2$, $e_5' = c_{13} e_5$. The product $e_2 e_3$ is eliminated and (184) takes the form

$$(185) \quad e_1 e_2 = e_4 = -e_2 e_1, \quad e_1 e_3 = e_5 = -e_3 e_1,$$

which is algebra 15 for $n = 5$. If in (184) $c_{13} = 0$, $c_{23} \neq 0$, we interchange e_1 and e_2 , replace e_4 by $-e_4$ and have the case which has just been considered. We observe that the algebra (185) is one of type B.

We shall now make a study of algebras of order n , degree 1, arising from an algebra of type B in $n - 1$ units or from a reducible algebra which is the direct sum of an algebra of type B in $n - 2$ units and a zero algebra of order 1.

We first consider algebras derived by means of Theorem 5 from

$$(186) \quad \begin{aligned} e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\ e_1 e_3 = e_{r+2} & & = -e_3 e_1, \\ \dots & & \dots, \\ e_{r-1} e_r = e_{n-1} & & = -e_r e_{r-1}, \end{aligned}$$

an algebra of type B in $n - 1$ units which has in the left members of its multiplication table all possible combinations $e_i e_j$ for $i \neq j$, $1 \leq r$, $j \leq r$. Then the number of rows in (186) is $n - 1 - r = rC_2 = r(r - 1)/2$ and hence

$$(187) \quad n = 1 + (r^2 + r)/2.$$

The new algebras take the form

$$(188a) \quad \begin{aligned} e_1 e_2 = e_{r+1} & & + c_{12} e_n = -e_2 e_1, \\ e_1 e_3 = e_{r+2} & & + c_{13} e_n = -e_3 e_1, \\ \dots & & \dots, \\ e_{r-1} e_r = e_{n-1} + c_{r-1, r} e_n & & = -e_r e_{r-1}, \end{aligned}$$

with

$$(188b) \quad \begin{array}{ll} e_1 e_{r+1} = & c_{1,r+1} e_n = -e_{r+1} e_n, \\ \cdot \cdot \cdot & \cdot \cdot \cdot, \\ e_{n-2} e_{n-1} = & c_{n-2,n-1} e_n = -e_{n-1} e_{n-2}. \end{array}$$

The set (188a) is composed of all products $e_i e_j$ for which $1 \leq r$, $j \leq r$, $i \neq j$, all possible combinations of $e_1, \dots, e_r$ taken two at a time. The set (188b) consists of all products $e_i e_j$ ($i, j = 1, \dots, n$; $i \neq j$) not included in (188a). Now put $e_{r+1}' = e_{r+1} + c_{12} e_n$, $e_{r+2}' = e_{r+2} + c_{13} e_n$, $\dots$, $e_{n-1}' = e_{n-1} + c_{r-1,r} e_n$. The algebras (188) become

$$(189a) \quad \begin{array}{lll} e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\ e_1 e_3 = e_{r+2} & & = -e_3 e_1, \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot, \\ e_{r-1} e_r = e_{n-1} & & = -e_r e_{r-1}, \end{array}$$

with

$$(189b) \quad \begin{array}{ll} e_1 e_{r+1} = & c_{1,r+1} e_n = -e_{r+1} e_1, \\ \cdot \cdot \cdot & \cdot \cdot \cdot, \\ e_{n-2} e_{n-1} = & c_{n-2,n-1} e_n = -e_{n-1} e_{n-2}. \end{array}$$

Let e_i, e_j, e_k be any three basal units such that $1 \leq r$, $j \leq r$, $k \leq r$, $i \neq j$, $i \neq k$, $j \neq k$. Then the products

$$(190) \quad e_i e_j = e_\ell = -e_j e_i, \quad e_i e_k = e_p = -e_k e_i, \quad e_j e_k = e_q = -e_k e_j$$

are in (189a), and $r+1 \leq \ell$, $p, q \leq n-1$. From (190) we find

$$(191) \quad \begin{array}{lll} e_i e_j e_k = e_\ell e_k = c_{\ell k} e_n = e_i e_q = c_{iq} e_n, \\ e_i e_k e_j = e_p e_j = c_{pj} e_n = -e_i e_q = -c_{iq} e_n, \\ e_k e_i e_j = e_k e_\ell = -c_{\ell k} e_n = -e_p e_j = -c_{pj} e_n, \end{array}$$

and therefore $c_{pj} = c_{\ell k} = c_{iq} = -c_{pj}$. Hence $c_{pj} = 0$. As $e_i e_k$ ranges over all the products in (189a), e_p ranges over all the basal units $e_{r+1}, \dots, e_{n-1}$. Then as e_j ranges, independently of e_i and e_k , over $e_1, \dots, e_r$, the product $e_p e_j$ ranges over all the products in (189b) having one subscript equal to or less than r and the other greater than r . Thus $e_p e_j = 0$ implies that every

product in (189b) in which one subscript is equal to or less than r and the other greater than r is zero. There remain in (189b) then only products in which both subscripts are greater than r . Consider any such product $e_p e_q$. As in (190), $e_p = e_1 e_k$. Hence $e_p e_q = e_1 e_k e_q = 0$ since $e_k e_q = 0$, having one subscript equal to or less than r and the other greater than r . Therefore the entire set (189b) is zero. Thus e_n is missing from (189) and the algebra is reducible.

We next consider algebras derived by means of Theorem 5 from the reducible algebra (189a) with n replaced by $n - 1$. The new algebras first assume the form

$$\begin{aligned}
 e_1 e_2 &= e_{r+1} & + c_{12} e_n &= -e_2 e_1, \\
 e_1 e_3 &= e_{r+2} & + c_{13} e_n &= -e_3 e_1, \\
 \dots & & \dots & , \\
 (192) \quad e_{r-1} e_r &= e_{n-2} + c_{r-1, r} e_n &= -e_r e_{r-1}, \\
 e_1 e_{r+1} &= c_{1, r+1} e_n &= -e_{r+1} e_r, \\
 \dots & & \dots & , \\
 e_{n-2} e_{n-1} &= c_{n-2, n-1} e_n &= -e_{n-1} e_{n-2}.
 \end{aligned}$$

The transformation $e_{r+1}' = e_{r+1} + c_{12} e_n$, $e_{r+2}' = e_{r+2} + c_{13} e_n$, ..., $e_{n-2}' = e_{n-2} + c_{r-1, r} e_n$ gives (192) the form

$$\begin{aligned}
 e_1 e_2 &= e_{r+1} & = -e_2 e_1, \\
 (193a) \quad e_1 e_3 &= e_{r+2} & = -e_3 e_1, \\
 \dots & & \dots , \\
 e_{r-1} e_r &= e_{n-2} & = -e_r e_{r-2},
 \end{aligned}$$

with

$$\begin{aligned}
 e_1 e_{r+1} &= c_{1, r+1} e_n &= -e_{r+1} e_1, \\
 (193b) \quad \dots & & \dots , \\
 e_{n-2} e_{n-1} &= c_{n-2, n-1} e_n &= -e_{n-1} e_{n-2}.
 \end{aligned}$$

As in the preceding case, let e_i, e_j, e_k be any three basal units with subscripts equal to or less than r , $i \neq j$, $i \neq k$, $j \neq k$. Then the products (190) are in (193a), but now $r + 1 \leq \ell, p, q \leq n - 2$. Then, as before, we can show that any product in (193b) in which one subscript is equal to or less than r and the other greater than r is zero, except those involving e_{n-1} . This unit does not occur in (193a). Now consider the products in (193b) in which both subscripts are greater than r . For $r + 1 \leq p, q \leq n - 2$, we have $e_p e_q = 0$ as in the preceding case. Also $e_p e_{n-1} = 0$ since $e_p e_{n-1} = e_i e_k e_{n-1} = c_{k,n-1} e_i e_n = 0$. Hence the only products not zero in (193b) are

$$e_i e_{n-1} = c_{i,n-1} e_n = -e_{n-1} e_i \quad (i = 1, \dots, r).$$

Now subject the basal units to the cyclic permutation

$(e_{r+1} e_{r+2} \dots e_{n-2} e_{n-1})$ and for convenience drop the second subscript of the $c_{i,n-1}$. The algebra (193) now has the form

$$(194) \quad \begin{aligned} e_1 e_2 &= e_{r+2} = -e_2 e_1, & e_1 e_3 &= e_{r+3} = -e_3 e_1, & \dots, \\ e_1 e_r &= e_{2r} = -e_r e_1, & e_2 e_3 &= e_{2r+1} = -e_3 e_2, & \dots, \\ e_2 e_r &= e_{3r-2} = -e_r e_2, & e_3 e_4 &= e_{3r-1} = -e_4 e_3, & \dots, \\ e_3 e_r &= e_{4r-5} = -e_r e_3, & \dots, & e_{r-1} e_r &= e_{n-1} = -e_r e_{r-1}, \\ e_i e_{r+1} &= c_i e_n = -e_{r+1} e_i & (i = 1, \dots, r). \end{aligned}$$

If all the $c_i = 0$ ($i = 1, \dots, r$), the algebra (194) is reducible. Hence we assume $c_1 \neq 0$ and eliminate the products $e_i e_{r+1}$ ($i = 2, \dots, r$) by means of the transformation

$$\begin{aligned} e_1' &= -c_1^{-1} c_1 e_1 + e_1 & (i = 2, \dots, r), \\ e_{2r+1}' &= c_1^{-1} (c_{2+1} e_{r+2} - c_2 e_{r+2+1}) + e_{2r+1} & (i = 1, \dots, r-2), \\ e_{3r-2+1}' &= c_1^{-1} (c_{3+1} e_{r+3} - c_3 e_{r+3+1}) + e_{3r-2+1} & (i = 1, \dots, r-3), \\ &\vdots \\ e_{n-1}' &= c_1^{-1} (c_r e_{2r-1} - c_{r-1} e_{2r}) + e_{n-1}. \end{aligned}$$

We now have (194) with

$$(195) \quad c_1 = 1, \quad c_i = 0 \quad (i = 2, \dots, r).$$

If in (194) $c_1 = 0$, $c_2 \neq 0$, we make the cyclic permutations $(e_1 e_2)$, $(e_{r+3} e_{2r+1})$, $(e_{r+4} e_{2r+2})$, $\dots$, $(e_{2r} e_{3r-2})$, replace e_{r+2} by $-e_{r+2}$ and have (194) with $c_1 \neq 0$. In case $c_1 = c_2 = 0$, $c_3 \neq 0$, the cyclic permutations $(e_1 e_3)$, $(e_{r+4} e_{3r-1})$, $(e_{r+5} e_{3r})$, $\dots$, $(e_{2r} e_{4r-5})$, $(-e_{r+2} e_{2r+1})$ and the replacing of e_{r+3} by $-e_{r+3}$ give us (194) with $c_1 \neq 0$. Other cases are treated similarly. Finally if $c_i = 0$ ($i = 1, \dots, r-1$), $c_r \neq 0$, make the permutations $(e_1 e_r)$, $(-e_{r+2} e_{3r-2})$, $(-e_{r+3} e_{4r-5})$, $(-e_{r+4} e_{5r-9})$, $\dots$, $(-e_{2r-1} e_{n-1})$ and replace e_{2r} by $-e_{2r}$. Then put $e_n' = c_r e_n$ and obtain (194) with (195). Now put $r = r' - 1$, that is, replace r by a new r which is equal to the original r increased by 1, so as to keep the algebra in the form (183). We now write the multiplication table in the form

$$(196) \quad \begin{array}{lll} e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\ e_1 e_3 = e_{r+2} & & = -e_3 e_1, \\ \dots & \dots\dots & \dots\dots, \\ e_{r-2} e_{r-1} = e_{n-1} & & = -e_{r-1} e_{r-2}, \\ e_1 e_r = e_n & & = -e_r e_1, \end{array}$$

where e_r occurs only in the last line.

Now consider the algebras derived from one of the form (196) in $n-1$ units by means of Theorem 5. The new algebras are first in the form

$$e_1 e_2 = e_{r+1} + c_{12} e_n = -e_2 e_1,$$

$$e_1 e_3 = e_{r+2} + c_{13} e_n = -e_3 e_1,$$

$$\dots \dots \dots \dots \dots,$$

$$e_{r-2} e_{r-1} = e_{n-2} + c_{r-2, r-1} e_n = -e_{r-1} e_{r-2},$$

$$e_1 e_r = e_{n-1} + c_{1r} e_n = -e_r e_1,$$

$$e_s e_t = c_{st} e_n = -e_t e_s$$

$$(s = 2, \dots, n-2; t = r, \dots, n-1; s < t),$$

but after the transformation $e_{r+1}' = e_{r+1} + c_{12} e_n$, $e_{r+2}' = e_{r+2} + c_{13} e_n$, ..., $e_{n-1}' = e_{n-1} + c_{1r} e_n$, they have the multiplication table

$$e_1 e_2 = e_{r+1} = -e_2 e_1,$$

$$e_1 e_3 = e_{r+2} = -e_3 e_1,$$

$$\dots \dots \dots \dots \dots,$$

$$(197) \quad e_{r-2} e_{r-1} = e_{n-2} = -e_{r-1} e_{r-2},$$

$$e_1 e_r = e_{n-1} = -e_r e_1,$$

$$e_s e_t = c_{st} e_n = -e_t e_s$$

$$(s = 2, \dots, n-2; t = r, \dots, n-1; s < t).$$

Let e_i, e_j, e_k be any three basal units such that $i \leq r-1$, $j \leq r-1$, $k \leq r-1$, $i \neq j$, $i \neq k$, $j \neq k$. Then we have (190) and (191) with $r+1 \leq \ell$, $p, q \leq n-2$, and $e_p e_j = 0$, that is, any product having one subscript less than r and the other greater than r and less than $n-1$ is zero. Also $e_p e_q = e_i e_k e_q = 0$ since $e_k e_q = 0$, and $e_p e_{n-1} = e_i e_k e_{n-1} = c_{k, n-1} e_i e_n = 0$. Thus any product with both subscripts greater than r is zero. Using the associative law we find $e_1 e_{n-1} = e_1^2 e_r = 0$, $e_1 e_{n-1} = -e_1 e_r e_1 = 0$ ($i = 2, \dots, r$), and $e_r e_1 = -e_r e_j e_k = 0$ ($i = r+1, \dots, n-1$; $j = 2, \dots, r-1$; $k = 1, \dots, r-2$), $e_r e_{n-1} = -e_r^2 e_1 = 0$. Hence the only products in the last line of (197) not zero are $e_1 e_r = c_{1r} e_n$ ($i = 2, \dots, r-1$). We now drop the second subscript

of c_{1r} and write the algebras in the form

$$\begin{aligned}
 (198) \quad & e_1 e_2 = e_{r+1} = -e_2 e_1, & e_1 e_3 = e_{r+2} = -e_3 e_1, & \dots, \\
 & e_1 e_{r-1} = e_{2r-2} = -e_{r-1} e_1, & e_2 e_3 = e_{2r-1} = -e_3 e_2, & \dots, \\
 & e_2 e_{r-1} = e_{3r-5} = -e_{r-1} e_2, & e_3 e_4 = e_{3r-4} = -e_4 e_3, & \dots, \\
 & e_3 e_{r-1} = e_{4r-9} = -e_{r-1} e_3, & \dots, & \\
 & e_{r-2} e_{r-1} = e_{n-2} = -e_{r-1} e_{r-2}, & e_1 e_r = e_{n-1} = -e_r e_1, & \\
 & e_1 e_r = c_1 e_n = -e_r e_1 & (i = 2, \dots, r-1). &
 \end{aligned}$$

If all the $c_i = 0$, (198) is reducible. Hence assume $c_2 \neq 0$ and put

$$\begin{aligned}
 e_1' &= -c_2^{-1} c_1 e_2 + e_1 & (i = 3, \dots, r-1), \\
 e_{r+1-1}' &= -c_2^{-1} c_1 e_{r+1} + e_{r+1-1} & (i = 3, \dots, r-1), \\
 e_{3r-4+1}' &= c_2^{-1} (c_{4+1} e_{2r-1} - c_3 e_{2r+1}) + e_{3r-4+1} & (i = 0, 1, \dots, r-5), \\
 e_{4r-8+1}' &= c_2^{-1} (c_{5+1} e_{2r} - c_4 e_{2r+1+1}) + e_{4r-8+1} & (i = 0, 1, \dots, r-6), \\
 & \dots \dots \dots \\
 e_{n-2}' &= c_2^{-1} (c_{r-1} e_{3r-6} - c_{r-2} e_{3r-5}) + e_{n-2}, \\
 e_n' &= c_2 e_n.
 \end{aligned}$$

This transformation eliminates all the c_i except c_2 and gives (198) the form

$$\begin{aligned}
 (199) \quad & e_1 e_2 = e_{r+1} & = -e_2 e_1, \\
 & e_1 e_3 = e_{r+2} & = -e_3 e_1, \\
 & \dots & \dots, \\
 & e_{r-2} e_{r-1} = e_{n-2} & = -e_{r-1} e_{r-2}, \\
 & e_1 e_r = e_{n-1} & = -e_r e_1, \\
 & e_2 e_r = e_n & = -e_r e_2,
 \end{aligned}$$

where e_r occurs only in the last two lines. If in (198) $c_2 = 0$, $c_3 \neq 0$, we make the permutations $(e_2 e_3)$, $(e_{2r} e_{3r-4})$, $(e_{2r+1} e_{3r-3})$, $\dots$, $(e_{3r-5} e_{4r-9})$, $(e_{r+1} e_{r+2})$, replace e_{2r-1} by $-e_{2r-1}$ and have

(198) with $c_2 \neq 0$. Other cases are treated similarly. Finally if in (198) $c_1 = 0$ ($i = 2, \dots, r - 2$), $c_{r-1} \neq 0$, make the permutations $(e_2 e_{r-1})$, $(e_{r+1} e_{2r-2})$, $(-e_{2r-1} e_{4r-9})$, $(-e_{2r} e_{5r-14})$, $\dots$, $(-e_{3r-6} e_{n-2})$, replace e_{3r-5} by $-e_{3r-5}$, put $e_n' = c_{r-1} e_n$ and obtain (199). We observe that (199) has one more row than the multiplication table of the algebra from which it is derived, (196) with n replaced by $n - 1$. This derivation of (199) from an algebra of the form (196) in $n - 1$ units is the case $v = 1$ of the following derivation.

Let the multiplication table of an algebra of type B in $n - 1$ units be

$$\begin{array}{lll}
 e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\
 e_1 e_3 = e_{r+2} & & = -e_3 e_1, \\
 \dots & \dots & \dots, \\
 (200) \quad e_{r-2} e_{r-1} = e_{n-v-1} & & = -e_{r-1} e_{r-2}, \\
 e_1 e_r = e_{n-v} & & = -e_r e_1, \\
 e_2 e_r = e_{n-v+1} & & = -e_r e_2, \\
 \dots & \dots & \dots, \\
 e_v e_r = e_{n-1} & & = -e_r e_v \quad (v < r - 1).
 \end{array}$$

The algebra derived from (200) by means of Theorem 5 has the form

$$\begin{array}{lll}
 e_1 e_2 = e_{r+1} & + c_{12} e_n = & -e_2 e_1, \\
 e_1 e_3 = e_{r+2} & + c_{13} e_n = & -e_3 e_1, \\
 \dots & \dots & \dots, \\
 e_v e_r = e_{n-1} & + c_{vr} e_n = & -e_r e_v, \\
 e_{v+1} e_r = c_{v+1,r} e_n & = & -e_r e_{v+1}, \\
 \dots & \dots & \dots, \\
 e_{n-2} e_{n-1} = c_{n-2,n-1} e_n & = & -e_{n-1} e_{n-2}.
 \end{array}$$

Now put $e_{r+1}' = e_{r+1} + c_{12} e_n$, $e_{r+2}' = e_{r+2} + c_{13} e_n$, $\dots$, $e_{n-1}' =$

$e_{n-1} + c_{vr}e_n$, and have the algebra in the form

$$\begin{aligned}
 (201a) \quad & \begin{array}{lll} e_1 e_2 = e_{r+1} & & = -e_2 e_1, \\ \dots & \dots & \dots, \\ e_{r-2} e_{r-1} = & e_{n-v-1} & = -e_{r-1} e_{r-2}, \\ e_1 e_r = & e_{n-v} & = -e_r e_1, \\ \dots & \dots & \dots, \\ e_v e_r = & e_{n-1} & = -e_r e_v. \end{array}
 \end{aligned}$$

with

$$\begin{aligned}
 (201b) \quad & \begin{array}{lll} e_{v+1} e_r = & c_{v+1,r} e_n = -e_r e_{v+1}, \\ \dots & \dots & \dots, \\ e_{r-1} e_r = & c_{r-1,r} e_n = -e_r e_{r-1}, \\ \dots & \dots & \dots, \\ e_{n-2} e_{n-1} = & c_{n-2,n-1} e_n = -e_{n-1} e_{n-2}. \end{array}
 \end{aligned}$$

Let e_i, e_j, e_k be any three basal units such that $i \leq r, j \leq r, k \leq r, i \neq j, i \neq k, j \neq k$. Then instead of (190) we have

$$\begin{aligned}
 (202) \quad & \begin{array}{ll} e_i e_j = -e_j e_i = e_\ell \text{ or } c e_n, & e_i e_k = -e_k e_i = e_p \text{ or } d e_n, \\ e_j e_k = -e_k e_j = e_q \text{ or } f e_n, & \end{array}
 \end{aligned}$$

where $r+1 \leq \ell, p, q, \leq n-1$. Then as i, j, k range independently over the numbers $1, \dots, r$, we have (191) in some cases while in others $e_i e_j e_k = c e_n e_k = 0$. But since $e_i e_j = -e_j e_i$ for all i and j , we have

$$(203) \quad e_i e_j e_k = -e_i e_k e_j = e_k e_i e_j.$$

Hence in view of (191) and (202) the products (203) are always zero and consequently $e_p e_j = 0$ as p ranges over the numbers $r+1, \dots, n-1$, and j over $1, \dots, r$. Also $e_p e_q = e_i e_k e_q = 0$. Thus all the products in (201b) with one subscript greater than r and one less than or equal to r are zero; also those having both

subscripts greater than r . Hence the only products in (201b) not zero are $e_1 e_r = c_{1r} e_n = -e_r e_1$ ($i = v + 1, \dots, r - 1$), and we now write (201) in the form

$$\begin{aligned}
 (204) \quad & e_1 e_2 = e_{r+1} = -e_2 e_1, \quad e_1 e_3 = e_{r+2} = -e_3 e_1, \quad \dots, \\
 & e_1 e_v = e_{r+v-1} = -e_v e_1, \quad e_1 e_{v+1} = e_{r+v} = -e_{v+1} e_1, \quad \dots, \\
 & e_1 e_{r-1} = e_{2r-2} = -e_{r-1} e_1, \quad e_2 e_3 = e_{2r-1} = -e_3 e_2, \quad \dots, \\
 & e_2 e_v = e_{2r+v-4} = -e_v e_2, \quad e_2 e_{v+1} = e_{2r+v-3} = -e_{v+1} e_2, \quad \dots, \\
 & e_2 e_{r-1} = e_{3r-5} = -e_{r-1} e_2, \quad \dots, \quad e_{r-2} e_{r-1} = e_{n-v-1} = -e_{r-1} e_{r-2}, \\
 & e_1 e_r = e_{n-v} = -e_r e_1, \quad \dots, \quad e_v e_r = e_{n-1} = -e_r e_v, \\
 & e_i e_r = c_{ir} e_n = -e_r e_i \quad (i = v + 1, \dots, r - 1).
 \end{aligned}$$

We shall use the following notation for certain products which may be in the algebra, but which are not indicated in (204).

$$\begin{aligned}
 e_{v+1} e_{v+2+1} &= e_{m+1} & (i = 0, 1, \dots, r - v - 4), \\
 e_{v+1} e_{r-1} &= e_{m+r-v-3} = e_s & (v + 1 < r - 1), \\
 e_{v+2} e_{v+2+1} &= e_{s+1} & (i = 1, \dots, r - v - 4), \\
 e_{v+2} e_{r-1} &= e_{s+r-v-3} = e_t & (v + 2 < r - 1), \\
 e_{v+3} e_{v+3+1} &= e_{t+1} & (i = 1, \dots, r - v - 5), \\
 e_{v+3} e_{r-1} &= e_{t+r-v-4} = e_w & (v + 3 < r - 1), \\
 e_{v+4} e_{v+5} &= e_{w+1} & (v + 4 < r - 1).
 \end{aligned}$$

If all the c_i are zero in (204) the algebra is reducible. Hence assume $c_{v+1} \neq 0$ and subject (204) to the transformation¹

¹It is evident that in some cases there may be overlapping in (205). For example, when $n = 26$, $r = 7$, $v = 3$, then $e_{n-v-1} = e_{22} = e_{s+1} = e_t$. In this case $r - v - 4 = 0$ and the expression for e_{t+1} holds only for $i = 0$.

$$\begin{aligned}
 e_{v+1}' &= -c_{v+1}^{-1} c_{v+1} e_{v+1} + e_{v+1} \quad (i = 2, \dots, r-v-1), \\
 e_{r+v-1+i}' &= -c_{v+1}^{-1} c_{v+1} e_{r+v} + e_{r+v-1+i} \quad (i = 2, \dots, r-v-1), \\
 e_{2r+v-4+i}' &= -c_{v+1}^{-1} c_{v+1} e_{2r+v-3} + e_{2r+v-4+i} \quad (i = 2, \dots, r-v-1), \\
 &\dots \dots \dots \\
 (205) \quad e_{s+1}' &= c_{v+1}^{-1} (c_{v+2+i} e_m - c_{v+2} e_{m+1}) + e_{s+1} \quad (i = 1, \dots, r-v-3), \\
 e_{t+1}' &= c_{v+1}^{-1} (c_{v+3+i} e_{m+1} - c_{v+3} e_{m+1+i}) + e_{t+1} \quad (i = 1, \dots, r-v-4), \\
 &\dots \dots \dots \\
 e_{n-v-1}' &= c_{v+1}^{-1} (c_{r-1} e_{s-1} - c_{r-2} e_s) + e_{n-v-1}, \\
 e_n' &= c_{v+1} e_n,
 \end{aligned}$$

which eliminates the c_i ($i = v+2, \dots, r-1$) and gives (204) the form

$$\begin{aligned}
 e_1 e_2 &= e_{r+1} & &= -e_2 e_1, \\
 &\dots \dots \dots & &\dots \dots \dots, \\
 (206) \quad e_{r-2} e_{r-1} &= e_{n-v-1} & &= -e_{r-1} e_{r-2}, \\
 e_1 e_r &= e_{n-v} & &= -e_r e_1, \\
 &\dots \dots \dots & &\dots \dots \dots, \\
 e_{v+1} e_r &= e_n & &= -e_r e_{v+1}.
 \end{aligned}$$

If in (204) $c_{v+1} = 0$, $c_{v+2} \neq 0$, we make the permutations $(e_{v+1} e_{v+2})$, $(e_{r+v} e_{r+v+1})$, $(e_{2r+v-3} e_{2r+v-2})$, $(e_{3r+v-7} e_{3r+v-6})$, $\dots$, $(e_{m+1} e_{s+1})$, $\dots$, $(e_s e_t)$, replace e_m by $-e_m$ and obtain (204) with $c_{v+1} \neq 0$. Other cases are treated similarly. Finally if $c_1 = 0$ ($i = v+1, \dots, r-2$), $c_{r-1} \neq 0$, we make the permutations $(e_{v+1} e_{r-1})$, $(e_{r+v} e_{2r-2})$, $(e_{2r+v-3} e_{3r-5})$, $(e_{3r+v-7} e_{4r-9})$, $\dots$, $(-e_m e_t)$, $(-e_{m+1} e_w)$, $\dots$, replace e_s by $-e_s$, put $e_n' = c_{r-1} e_n$ and obtain (206).

The algebra (206) has one more row in its multiplication table than (200), the algebra from which it is derived. By an induction on v it is seen that this process of increasing the

number of rows in the multiplication table continues until the number of rows is $n - r = {}_rC_2 = (r^2 - r)/2$. Then we have an algebra of the form (186) with $n - 1$ replaced by n , for which (187), with the same replacement, holds. Then, as we have seen, the next algebra, derived from this one, is reducible. Keeping this fact in mind and the fact that r was replaced by $r - 1$ in the derivation of algebra (196) from the reducible algebra (189a) we see that the values of n for which algebras of type B exist are given by the formula $1 + [(r - 1)^2 + (r - 1)]/2 < n \leq (r^2 + r)/2$. Hence we have proved

THEOREM 10. The values of n for which algebras of type B exist are given by

$$(207) \quad 1 + (r^2 - r)/2 < n \leq (r^2 + r)/2,$$

where r is the number of units e_1 involved in the products e_1e_j in the left members of the multiplication table, and hence $n - r$ is the number of rows in the table.

For $r = 2$, the formula (207) gives $2 < n \leq 3$ and $n = 3$. For $r = 3$, $4 < n \leq 6$, $n = 5, 6$. For $r = 4$, $7 < n \leq 10$, $n = 8, 9, 10$.

We shall now show that an algebra N of type B with all the basal units $e_1, \dots, e_n$ present is irreducible. The discussion in this section, with an induction on n , shows that any such algebra has a multiplication table of the form

If $e_1'e_2' \neq 0$ then some $a_{12j} \neq 0$. Set $\sum_{i=r+1}^n a_{12i}e_i' = e_j''$. If $\sum_{i=r+1}^n a_{13i}e_i'$ and $\sum_{i=r+1}^n a_{12i}e_i'$ are linearly dependent, $\sum_{i=r+1}^n a_{13i}e_i' = me_j''$. If they are independent, set $\sum_{i=r+1}^n a_{13i}e_i' = e_k''$ where k is one of $r+1, \dots, n$ but $k \neq j$. Continue this process until each sum $\sum_{i=r+1}^n a_{sti}e_i'$ different from zero is replaced by a single term $m_j e_j''$ where j is one of $r+1, \dots, n$. Then the number of units e_j'' ($j = r+1, \dots, n$) in the multiplication table will be at most equal to the number of rows in (208). If $p = 1$, N has the multiplication table (208b) with $p = 1$, the number of rows in the table is at most $r-1C_2 = (r-1)(r-2)/2$, and hence the number of e_j'' ($j = r+1, \dots, n$) in the table is less than $n - r$. Therefore we cannot have $B_1 = (e_1', \dots, e_r')$, $B_2 = (e_{r+1}', \dots, e_n')$. Hence we assume $p > 1$ and similarly $r - p > 1$. Then $p(r - p - 1) > (r - p - 1)$, $p(r - p) > r - 1$, $p^2 - rp < 1 - r$. Then $2p^2 - 2pr < 2 - 2r$ and $p^2 - p + r^2 - 2rp + p^2 - r + p < r^2 - 3r + 2$. Hence $p(p-1)/2 + (r-p)(r-p-1)/2 < (r-1)(r-2)/2$, that is, $pC_2 + r-pC_2 < r-1C_2$, and obviously $pC_2 + r-pC_2 < n - r$. Hence the number of e_j'' ($j = r+1, \dots, n$) in the multiplication table is less than $n - r$ and we cannot have $B_1 = (e_1', \dots, e_r')$, $B_2 = (e_{r+1}', \dots, e_n')$. Therefore N is irreducible.

3. Algebras of order 5. We are now able to classify algebras in 5 basal units. Those of degrees 5, 4 and 3 are found by setting $n = 5$ in algebras 1-7 in n units. Those of degree 2 are algebras 8, 9, 10, 11, 12, 13. Algebras 8 and 9 are the general algebras 8 and 9 with $n = 5$. Algebras 10 and 11 are (90) and (91) with $n = 5$. Algebras 12 and 13 were derived in Chapter V, Section 8. By Theorems 9 and 10 in the present chapter there is an algebra of type A and one of type B. They are algebras 14 and 15 respectively.

CHAPTER VII

SUMMARY AND CLASSIFICATION

Our results enable us to classify all linear associative nilpotent algebras of orders less than 6; and those of any order n of degrees n , $n - 1$, $n - 2$, $n - 3$, according to the degree and the linear sets B_1 defined in the Introduction. Such a classification is given in the following pages. It is to be remembered that in general the algebras listed are in reality classes of algebras.

We have also found that the multiplication table of a regular algebra of any order and any degree may be given in one of the forms (52) and (53). We observe that if N is a regular algebra, the centrum is always N^2 . In the classification of algebras of order n the regular algebras are seen to be those numbered 1, 2, 3, 4, 5, 8, 9. In particular, the algebras of order 3 are all regular except algebra 4; those of order 4 are regular except algebras 6 and 7; while the regular algebras of order 5 are algebras 1, 2, 3, 4, 5, 8, 9.

Two types of algebras of degree 1 have been generalized, types A and B, discussed in Chapter VI.

ALGEBRAS OF ORDER 3

I. Degree 3.

1. $B_1 = (e_1)$, $B_2 = (e_2)$, $B_3 = (e_3)$.

a. Commutative.

Algebra 1. $e_1^i = e_1$ ($i = 1, 2, 3$).

II. Degree 2.

$$1. B_1 = (e_1, e_2), B_2 = (e_3).$$

a. Commutative.

$$\text{Algebra 2. } e_1^2 = ae_3, e_2^2 = e_3 \ (a \neq 0).$$

b. Non-commutative.

$$\text{Algebra 3. } e_1^2 = ae_3, e_2e_1 = e_3 = e_2^2.$$

III. Degree 1.

$$1. B_1 = (e_1, e_2), B_2 = (e_3).$$

b. Non-commutative.

$$\text{Algebra 4. } e_1e_2 = e_3 = -e_2e_1.$$

ALGEBRAS OF ORDER 4

I. Degree 4.

$$1. B_1 = (e_1), B_2 = (e_2), B_3 = (e_3), B_4 = (e_4).$$

a. Commutative.

$$\text{Algebra 1. } e_i^1 = e_i \ (i = 1, 2, 3, 4).$$

II. Degree 3.

$$1. B_1 = (e_1, e_2), B_2 = (e_3), B_3 = (e_4).$$

a. Commutative.

$$\begin{aligned} \text{Algebra 2. } e_1^2 &= ae_4 \ (a \neq 0), \\ e_2^{i-1} &= e_1 \ (i = 2, 3, 4). \end{aligned}$$

b. Non-commutative.

$$\begin{aligned} \text{Algebra 3. } e_1^2 &= ae_4, e_2e_1 = e_4, \\ e_2^{i-1} &= e_1 \ (i = 2, 3, 4). \end{aligned}$$

III. Degree 2.

$$1. B_1 = (e_1, e_2, e_3), B_2 = (e_4).$$

a. Commutative.

$$\begin{aligned} \text{Algebra 4. } e_1^2 &= ae_4, e_2^2 = be_4 \ (ab \neq 0), \\ e_3^2 &= e_4. \end{aligned}$$

b. Non-commutative.

$$\text{Algebra 5. } e_1^2 = ae_4, \quad e_2^2 = be_4, \quad e_3^2 = e_4, \\ e_1e_2 = e_4, \quad e_2e_1 = ce_4, \quad e_3e_2 = e_4.$$

$$2. B_1 = (e_1, e_2), B_2 = (e_3, e_4).$$

a. Commutative.

$$\text{Algebra 6. } e_1^2 = be_4 \quad (b \neq k^2, k \text{ in } F), \\ e_1e_2 = e_3 = e_2e_1, \quad e_2^2 = e_4.$$

b. Non-commutative.

$$\text{Algebra 7. } e_1^2 = ae_3 + be_4, \quad e_2^2 = e_4, \\ e_1e_2 = e_3, \quad e_2e_1 = de_3 + e_4.$$

ALGEBRAS OF ORDER 5

I. Degree 5.

$$1. B_1 = (e_1), B_2 = (e_2), B_3 = (e_3), B_4 = (e_4), B_5 = (e_5).$$

a. Commutative.

$$\text{Algebra 1. } e_1^i = e_1 \quad (i = 1, \dots, 5).$$

II. Degree 4.

$$1. B_1 = (e_1, e_2), B_2 = (e_3), B_3 = (e_4), B_4 = (e_5).$$

a. Commutative.

$$\text{Algebra 2. } e_1^2 = ae_5 \quad (a \neq 0), \\ e_2^{i-1} = e_1 \quad (i = 2, 3, 4, 5).$$

b. Non-commutative.

$$\text{Algebra 3. } e_1^2 = ae_5, \quad e_2e_1 = e_5, \\ e_2^{i-1} = e_1 \quad (i = 2, 3, 4, 5).$$

III. Degree 3.

$$1. B_1 = (e_1, e_2, e_3), B_2 = (e_4), B_3 = (e_5).$$

a. Commutative.

$$\text{Algebra 4. } e_1^2 = ae_5, \quad e_2^2 = be_5 \quad (ab \neq 0), \\ e_3^{i-2} = e_1 \quad (i = 3, 4, 5).$$

b. Non-commutative.

Algebra 5. $e_1^2 = ae_5, e_2^2 = be_5,$

$$e_1e_2 = e_5, e_2e_1 = ce_5, e_3e_2 = e_5,$$

$$e_3^{1-2} = e_1 \quad (i = 3, 4, 5).$$

2. $B_1 = (e_1, e_2), B_2 = (e_3, e_4), B_3 = (e_5).$

a. Commutative.

Algebra 6. $e_1^2 = ae_3 + cge_4 + be_5,$

$$e_1e_2 = ce_3 = e_2e_1, e_1e_3 = ge_5 = e_3e_1,$$

$$e_2^{1-2} = e_1 \quad (i = 4, 5),$$

(a and c not both zero).

b. Non-commutative.

Algebra 7. $e_1^2 = ae_3 + cge_4 + be_5,$

$$e_1e_2 = ce_3, e_2e_1 = de_3 + cke_4 + fe_5,$$

$$e_1e_3 = ge_5, e_3e_1 = he_5, e_2e_3 = ke_5,$$

$$e_4e_1 = k(c + d)e_5, e_2^{1-2} = e_1 \quad (i = 4, 5),$$

(a, c, d not all zero).

IV. Degree 2.

1. $B_1 = (e_1, e_2, e_3, e_4), B_2 = (e_5).$

a. Commutative.

Algebra 8. $e_1^2 = c_1e_5 \quad (i = 1, 2, 3; c_1 \neq 0),$

$$e_4^2 = e_5.$$

b. Non-commutative.

Algebra 9. $e_1e_j = c_{1j}e_5 \quad (i, j = 1, 2, 3),$

$$e_4e_3 = e_5, e_4^2 = e_5.$$

2. $B_1 = (e_1, e_2, e_3), B_2 = (e_4, e_5).$

a. Commutative.

Algebra 10. $e_1^2 = ae_4 + be_5, e_2^2 = he_4 + je_5,$

$$e_1e_2 = ce_4 + de_5 = e_2e_1,$$

$$e_2e_3 = me_4 = e_3e_2, e_3^2 = e_5.$$

b. Non-commutative.

Algebra 11. $e_1^2 = ae_4 + be_5, e_2^2 = he_4 + je_5,$
 $e_1e_2 = ce_4 + de_5, e_2e_1 = fe_4 + ge_5,$
 $e_3e_1 = ke_4 + le_5,$
 $e_2e_3 = me_4, e_3e_2 = pe_4 + qe_5,$
 $e_3^2 = e_5.$

3. $B_1 = (e_1, e_2), B_2 = (e_3, e_4), B_3 = (e_5).$

a. Commutative.

Algebra 12. $e_1^2 = ae_3 + be_4 + ce_5,$
 $e_1e_2 = de_3 + fe_5 = e_2e_1,$
 $e_1e_3 = ke_5 = e_3e_1, e_1e_4 = dpe_5 = e_4e_1,$
 $e_2e_3 = pe_5 = e_3e_2, e_2^2 = e_4,$
 $(a \text{ and } d \text{ not both zero; } ak + bdp = 0).$

b. Non-commutative.

Algebra 13. $e_1^2 = ae_3 + be_4 + ce_5, e_2^2 = e_4,$
 $e_1e_2 = de_3 + fe_5, e_2e_1 = ge_3 + he_4 + je_5,$
 $e_1e_3 = ke_5, e_3e_1 = me_5, e_1e_4 = dre_5,$
 $e_2e_3 = pe_5, e_3e_2 = re_5, e_4e_1 = gpe_5,$
 $(a, d, g \text{ not all zero; } ak + bdr = 0 = am + bgp).$

V. Degree 1.

1. $B_1 = (e_1, e_2, e_3, e_4), B_2 = (e_5).$

b. Non-commutative.

Algebra 14. $e_1e_2 = e_5 = -e_2e_1,$
 $e_3e_4 = e_5 = -e_4e_3.$

2. $B_1 = (e_1, e_2, e_3), B_2 = (e_4, e_5).$

b. Non-commutative.

Algebra 15. $e_1e_2 = e_4 = -e_2e_1,$
 $e_1e_3 = e_5 = -e_3e_1.$

CERTAIN ALGEBRAS OF ORDER n I. Degree n .

$$1. B_1 = (e_1), B_2 = (e_2), \dots, B_n = (e_n).$$

a. Commutative.

$$\text{Algebra 1. } e_1^i = e_1 \quad (i = 1, \dots, n).$$

II. Degree $n - 1$ ($n > 2$).

$$1. B_1 = (e_1, e_2), B_2 = (e_3), B_3 = (e_4), \dots, B_{n-1} = (e_n).$$

a. Commutative.

$$\begin{aligned} \text{Algebra 2. } e_1^2 &= ae_n \quad (a \neq 0), \\ e_2^{i-1} &= e_1 \quad (i = 2, \dots, n). \end{aligned}$$

b. Non-commutative.

$$\begin{aligned} \text{Algebra 3. } e_1^2 &= ae_n, \quad e_2 e_1 = e_n, \\ e_2^{i-1} &= e_1 \quad (i = 2, \dots, n). \end{aligned}$$

III. Degree $n - 2$. The forms of algebras 4 and 5 hold for $n > 3$; those of algebras 6 and 7, for $n > 4$.

$$1. B_1 = (e_1, e_2, e_3), B_2 = (e_4), B_3 = (e_5), \dots, B_{n-2} = (e_n).$$

a. Commutative.

$$\begin{aligned} \text{Algebra 4. } e_1^2 &= ae_n, \quad e_2^2 = be_n \quad (ab \neq 0), \\ e_3^{i-2} &= e_1 \quad (i = 3, \dots, n). \end{aligned}$$

b. Non-commutative.

$$\begin{aligned} \text{Algebra 5. } e_1^2 &= ae_n, \quad e_2^2 = be_n, \\ e_1 e_2 &= e_n, \quad e_2 e_1 = ce_n, \quad e_3 e_2 = e_n, \\ e_3^{i-2} &= e_1 \quad (i = 3, \dots, n). \end{aligned}$$

$$2. B_1 = (e_1, e_2), B_2 = (e_3, e_4), B_3 = (e_5), \dots, B_{n-2} = (e_n)$$

a. Commutative.

$$\begin{aligned} \text{Algebra 6. } e_1^2 &= ae_3 + ce_{n-1} + be_n, \\ e_1 e_2 &= ce_3 = e_2 e_1, \quad e_1 e_3 = ge_n = e_3 e_1, \\ e_2^{i-2} &= e_1 \quad (i = 4, \dots, n), \\ &(a \text{ and } c \text{ not both zero}). \end{aligned}$$

b. Non-commutative.

$$\text{Algebra 7. } e_1^2 = ae_3 + cge_{n-1} + be_n,$$

$$e_1e_2 = ce_3, \quad e_2e_1 = de_3 + cke_{n-1} + fe_n,$$

$$e_1e_3 = ge_n, \quad e_3e_1 = he_n, \quad e_2e_3 = ke_n,$$

$$e_4e_1 = k(c + d)e_n,$$

$$e_2^{1-2} = e_1 \quad (i = 4, \dots, n),$$

$$(a, c, d \text{ not all zero}).$$

IV. Degree $n - 3$. The forms of algebras 8 and 9 hold for $n > 4$; those of algebras 10 and 11 for $n > 5$; those of algebras 12 and 13 for $n > 7$.

$$1. B_1 = (e_1, e_2, e_3, e_4), B_2 = (e_5), B_3 = (e_6), \dots, B_{n-3} = (e_n).$$

a. Commutative.

$$\text{Algebra 8. } e_1^2 = c_1e_n \quad (i = 1, 2, 3; c_1 \neq 0),$$

$$e_4^{1-3} = e_1 \quad (i = 4, \dots, n).$$

b. Non-commutative.

$$\text{Algebra 9. } e_1e_j = c_{1j}e_n \quad (i, j = 1, 2, 3),$$

$$e_4e_3 = e_n, \quad e_4^{1-3} = e_1 \quad (i = 4, \dots, n).$$

$$2. B_1 = (e_1, e_2, e_3), B_2 = (e_4, e_5), B_3 = (e_6), B_4 = (e_7), \\ \dots, B_{n-3} = (e_n).$$

a. Commutative.

$$\text{Algebra 10. } e_1^2 = ae_4 + be_n,$$

$$e_2^2 = he_4 + mte_{n-1} + je_n,$$

$$e_1e_2 = ce_4 + de_n = e_2e_1,$$

$$e_2e_3 = me_4 = e_3e_2, \quad e_1e_4 = re_n = e_4e_1,$$

$$e_2e_4 = te_n = e_4e_2,$$

$$e_3^{1-3} = e_1 \quad (i = 5, \dots, n).$$

b. Non-commutative.

Algebra 11.

$$\begin{aligned}
 e_1^2 &= ae_4 + be_n, & e_2^2 &= he_4 + mte_{n-1} + je_n, \\
 e_1e_2 &= ce_4 + mre_{n-1} + de_n, & e_2e_1 &= fe_4 + ge_n, \\
 & & e_3e_1 &= ke_4 + le_n, \\
 e_2e_3 &= me_4, & e_3e_2 &= pe_4 + mwe_{n-1} + qe_n, \\
 e_1e_4 &= re_n, & e_4e_1 &= se_n, & e_2e_4 &= te_n, & e_4e_2 &= ve_n, \\
 e_3e_4 &= we_n, & e_5e_1 &= wke_n, & e_5e_2 &= w(m+p)e_n, \\
 e_3^{1-3} &= e_1 \quad (i = 5, \dots, n).
 \end{aligned}$$

3. $B_1 = (e_1, e_2)$, $B_2 = (e_3, e_4)$, $B_3 = (e_5, e_6)$, $B_4 = (e_7)$,
 $\dots$, $B_{n-3} = (e_n)$.

a. Commutative. For $n = 6$, replace e_{n-1} by e_4 , omit the term $dpse_{n-2}$ in e_1^2 and dwe_{n-1} in e_1e_3 . For $n = 7$, replace e_{n-2} by e_4 .

Algebra 12.

$$\begin{aligned}
 e_1^2 &= ae_3 + be_5 + dpse_{n-2} + (dl + fs)e_{n-1} + ce_n, \\
 e_1e_2 &= de_3 + fe_5 = e_2e_1, \\
 e_1e_3 &= ke_5 + dwe_{n-1} + le_n = e_3e_1, \\
 e_2e_3 &= pe_5 = e_3e_2, & e_1e_4 &= dpe_5 = e_4e_1, \\
 e_1e_5 &= se_n = e_5e_1, & e_3^2 &= we_n, & e_2^2 &= e_4, \\
 e_2^{1-3} &= e_1 \quad (i = 6, \dots, n), \\
 &(a \text{ and } d \text{ not both zero}).
 \end{aligned}$$

b. Non-commutative. For $n = 6$, replace e_{n-1} by e_4 , omit $drse_{n-2}$ in e_1^2 , $drue_{n-2}$ in e_2e_1 and the terms in e_{n-1} in the products e_1e_3 , e_3e_1 , e_2e_3 , e_4e_1 , omit e_6e_1 and write $e_4e_3 = upe_6$. For $n = 7$, replace e_{n-2} by e_4 in the products e_1^2 and e_2e_1 .

Algebra 13.

$$\begin{aligned}
e_1^2 &= ae_3 + be_5 + drse_{n-2} + (d\ell + fs)e_{n-1} + ce_n, \\
e_1e_2 &= de_3 + fe_5, \\
e_2e_1 &= ge_3 + he_5 + drue_{n-2} + (dq + fu)e_{n-1} + je_n, \\
e_1e_3 &= ke_5 + rse_{n-1} + \ell e_n, \\
e_3e_1 &= me_5 + dwe_{n-1} + ve_n, \\
e_2e_3 &= pe_5 + rue_{n-1} + qe_n, & e_3e_2 &= re_5, \\
e_1e_4 &= dre_5, & e_4e_1 &= gpe_5 + ru(d + g)e_{n-1} \\
& & & + [q(d + g) + u(f + h)]e_n, \\
e_1e_5 &= se_n, & e_5e_1 &= te_n, & e_2e_5 &= ue_n, \\
e_4e_3 &= u(p + r)e_n, & e_6e_1 &= u(gp + dr + gr)e_n, \\
e_3^2 &= we_n, & e_2^2 &= e_4, \\
e_2^{i-3} &= e_1 \quad (i = 6, \dots, n), \\
& (a, d, g \text{ not all zero}).
\end{aligned}$$

REGULAR ALGEBRAS OF ORDER n , DEGREE $n - r + 1$

$$B_1 = (e_1, \dots, e_r), B_2 = (e_{r+1}), B_3 = (e_{r+2}), \dots, B_{n-r+1} = (e_n).$$

a. Commutative.

$$\begin{aligned}
(52) \quad e_1^2 &= c_1e_n & (i = 1, \dots, r-1; c_1 \neq 0), \\
e_1^{i-r+1} &= e_1 & (i = r, \dots, n).
\end{aligned}$$

b. Non-commutative.

$$\begin{aligned}
(53) \quad e_ie_j &= c_{ij}e_n & (i, j = 1, \dots, r-1), \\
ee_{r-1} &= e_n, & e_1^{i-r+1} &= e_1 & (i = r, \dots, n).
\end{aligned}$$

VITA

Fannie Wilson Boyce was born near Lentner, Missouri, on March 16, 1897. When four years of age she moved with her parents to Colorado where she received her grade school education. Her high school work was done in the academy of John Fletcher College in University Park, Iowa. In 1918 she received the degree Bachelor of Arts from Fletcher College. After teaching two years she spent a year in Penn College from which she was graduated in 1921 with the degree B.A. She then taught mathematics in the academy of Olivet College one year and in the academy of Marion College five years. The next year was spent in study at the University of Wisconsin where she took courses under Professors Van Vleck, Skinner and Ingraham and received the degree Master of Arts in 1928. After teaching in the high school in Platteville, Wisconsin, and in the high school and junior college in La Salle, Illinois, she began teaching in Wheaton College at Wheaton, Illinois, in 1930. She was granted leave of absence for the years 1933 to 1934 and 1936 to 1937 and spent these two years and the summers of 1933, 1934, 1935, 1937 in study at the University of Chicago. She attended courses given by Professors Albert, Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon and Zachariasen. She wishes to acknowledge her indebtedness to all of these, and especially to Professor Albert, under whose guidance this paper was written.

